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FREE FALLING PARTICLE IN DENSITY STRATIFIED FLUID

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ABSTRACT

Free falling particle due to gravity in the linear density stratified fluid was investigated theoretically and experimentally. The equations governing the particle motion in a homogeneous viscous fluid can be applied to this case if the density stratified fluid body is considered to be composed of infinite homogeneous layers of finite thickness. The particle motion is found to be a function of the fluid density gradient. The effect of the density gradient is very pronounced for the particle-fluid density ratio almost equal to one. For high particle-fluid density ratio, the density gradient effect is not appreciable. The proposed calculation of particle displacement can well describe the phenomena of the particle sedimentation in the zones of thermoclines in most of the natural water bodies.

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CHAPTER I- INTRODUCTION

I-1 General Description

Solid particles settling in the environment of density stratified fluid is one of the important sedimentation phenomena. Ocean, lake and reservoir fluid are commonly known to be stratified due to the spatial differences of density, salinity and temperature. Particle sedimentation in such a non-homogeneous fluid is different from that in the homogeneous fluid. An investigation on the motion of a particle free falling in the density stratified fluid will give the fundamental principle for the analysis of many complex sedimentation problems. Examples to show the widespread applications of this type of particle motion in stratified fluid are: sedimentation problems and density currents in lakes and reservoirs, deposition of marine sediment, settling of sediment in estuaries and river mouths. These problems are closely related to the water resources management and the water pollution control.

I-2 Historical Studies and Scope of This Investigation

Accelerated and steady state motion of a spherical particle in a viscous fluid of infinite extent has been studied extensively. The results are well presented by Brush et al (1), Hjelmfelt and Mockros (4). However, the studies are for constant fluid density with respect to time and space. It is expected that the particle motion would behave differently if the density of the ambient fluid is changed resulting from the density stratification. This report will present the study of the motion of a spherical particle falling freely due to the gravitational force in a linear density stratified fluid.

Most density stratification in the nature water bodies can be classified in two categories, the two-layer system and the system of linear density stratification. The density stratification will receive further discussion in chapter 2. The two-layer system consists of two homogeneous density zones. Particles settling in such an environment can be analysed using the results from the previous studies by treating the particle motion within each individual homogeneous zone. For the other case of linear stratified fluid, density increased linearly in depth, analyses is made through the following concept.

Consider a linearly stratified water body to be composed of an infinite number of very thin layers. Each layer has constant density and thickness in the same order of magnitude as the size of the particle.

Then, the previously developed theories hold in each layer. In other words, the density parameter appearing in the governing equation of motion for the particle can be considered as constant locally in time or space. Of course, this is an approximation. However, the results of this study will show a very good agreement between the analytical results and the experimental data.

CHAPTER II- THEORETICAL CONSIDERATIONS

II-1 Equation of Motion for the Free Falling Particle in Homogeneous Viscous Fluid

The particle is assumed to be spherical with diameter d and density ρ_s. The density and the dynamic viscosity of the fluid are ρ and μ. Denoting Ms and M as the mass of the sphere and the mass of the fluid displaced by the sphere, the equation of the unsteady motion for a free falling spherical particle in a viscous fluid of infinite extent is (6).

$$\left(m_s+\frac{m}{2}\right)\frac{dV}{dt}+3\pi\mu d V+\frac{3}{2}d^2\sqrt{\pi\rho\mu}\int_0^t \frac{\frac{dV}{d\tau}}{\sqrt{t-\tau}}\,d\tau=(m_s-m)g \quad \cdots\cdots(2\text{-}1)$$

In which, V is the particle velocity; t is the time; τ is a time interval less than t; and g is the acceleration due to gravity. In this relation, the first term is the mass times the acceleration of the sphere; the second term is the acceleration of the fluid added mass. The third term is the steady state drag. The fourth term is Basset history term which corrects the viscous drag for the transient condition. The last term is the buoyant force. The added mass coefficient is assumed to remain equal to 1/2 for the sphere moving at the time of acceleration.

Equation (2-1) is based on neglecting the convective acceleration in the Navier-Stokes equation and considering the drag force to be linear. For particle motion outside Stokes range, the resistant force is no longer a linear function of the particle velocity V. The drag is a function of the drag coefficient C_D. If C_D is in terms of the instantaneous Reynolds number $\mathbb{R}$, based on the instantaneous particle velocity, the equation for the particle motion above the Stokes range becomes

$$\left(m_s + \frac{m}{2}\right)\frac{dV}{dt} + 3K\pi\mu V d + \frac{3}{2}d^2\sqrt{\pi\rho\mu}\int_0^t \frac{\frac{dV}{d\tau}}{\sqrt{t-\tau}}\,d\tau = (m_s - m)g \qquad (2\text{-}2)$$

in which

$$3K\pi\mu V d = C_D\frac{\pi d^2}{8}|V|\cdot V \qquad (2\text{-}3)$$

where K is a correction coefficient in terms of an empirical expression for the convenience of digital computer calculations. (7)

$$K = \frac{1}{6}\left(0.75\,\mathbb{R}^{0.72} + 6.0\right) \qquad (2\text{-}4)$$

and

$$\mathbb{R} = \frac{V\rho d}{\mu} \qquad (2\text{-}5)$$

II-2 Equation of Motion for the Free Falling Particle in Density Stratified Fluid

As can be seen in Equation (2-1) and Equation (2-2), the velocity of the particle is a function of the variable of time. All other parameters are constant with respect to time. In a density stratified fluid, the density of the fluid is a function of space only. Hence the equations can be solved by holding the fluid density to be constant with respect to time. For the present problem, the particle motion is in the gravitational direction along which the density is stratified. Thus, the two governing equations in Section II-1 remain to be the same equations of motion for a free falling particle in density stratified fluid. It is noted from the governing equations that the density field of the fluid has effect on the particle motion through the terms of added mass, Basset history, and buoyant force.

II-3 Density Stratified Fluid

In general, a nature water body such as lake & ocean possesses a density field in common (2). The upper zone is called epilimnion in which the density is fairly constant. The lower zone is the hypolimnion which is also of nearly constant density and greater than the epilimnion. The transition between the two zones is so called the thermocline. This zone is thin in depth compared with the other two zones. The density gradient in the thermocline varies from very steep to very mild. Numerous thermoclines are nearly linear in temperature and density stratification. In effect the typical stratified environment consists of zones of either constant density or linearly stratified zones.

The constant density zones can be considered as homogeneous fluid. The thermoclines are considered as non-homogeneous field increasing in density linearly with depth. The former cases are described by $\rho =$ constant. The thermocline is represented by the following equation.

$$\rho = \rho_o - \frac{d\rho}{dh}(Y-h) = \rho_o - G(Y-h) \qquad (2\text{-}6)$$

In which ρ is the density at any depth h measuring from the top of the thermocline. ρ_o is the density at the bottom of the thermocline. G is the density gradient. Y is the thickness of the thermocline. If the thermocline is extremely steep, the density environment becomes a two-layered field.

The particle motion in two-layered field can be resolved into particle motion in two separated zones of homogeneous field. To study the particle motion in a linear density stratified field is the main interest of this research. Thus, one will be able to describe the particle settling due to the gravity in most of the nature water bodies with the complete information of particle motion in both homogeneous and linealy stratified density field.

II-4 Solution for Particle Velocity within the Stokes Range

Applying a Laplace transform with L {V (t)} = f (s) and V (t) = V (0) = 0, equation (2-1) may be transformed into a rearranged form as

$$f(s)=\left[\frac{\left(\frac{\rho_s}{\rho}-1\right)g}{\left(\frac{\rho_s}{\rho}+1\right)}\right]\left[\frac{\frac{-(a+b)}{a^2b^2}}{\sqrt{s}}+\frac{\frac{1}{ab}}{s}-\frac{\frac{1}{a^2(a-b)}}{\sqrt{s}+a}+\frac{\frac{1}{b^2(a-b)}}{\sqrt{s}+b}\right] \qquad (2\text{-}7)$$

in which

$$a,b = \frac{9\sqrt{\nu}}{2\left(\frac{\rho_s}{\rho}+\frac{1}{2}\right)d} \pm \frac{3}{2d}\sqrt{\frac{9\nu}{\left(\frac{\rho_s}{\rho}+\frac{1}{2}\right)^2} - \frac{8\nu}{\left(\frac{\rho_s}{\rho}+\frac{1}{2}\right)}} \qquad (2\text{-}8)$$

The inverse transform gives the solution, after simplification, as

$$V = \frac{\left(\frac{\rho_s}{\rho}-1\right)g}{\left(\frac{\rho_s}{\rho}+\frac{1}{2}\right)}\left[\frac{1}{ab} + \frac{1}{a(a-b)}e^{a^2 t}\,\mathrm{erfc}(a\sqrt{t}) - \frac{1}{b(a-b)}e^{b^2 t}\,\mathrm{erfc}(b\sqrt{t})\right] \qquad (2\text{-}9)$$

in which

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-y^2}\,dy = 1 - \mathrm{erf}(x) \qquad (2\text{-}10)$$

where y is the dummy variable and x is the argument of the error function. As $t \longrightarrow \infty$, it may be shown, by L´ Hospital´s rule, that the particle approaches its "local" terminal fall velocity V_T.

$$V = V_T = \frac{\left(\frac{\rho_s}{\rho}-1\right)g d^2}{18\nu} \qquad (2\text{-}11)$$

The term of terminal fall velocity is a commonly used terminology in the field of sedimentation. It is referred to the homogeneous density field because all parameters in the expression are not function of time anymore as $t \longrightarrow \infty$. However, for the case of linear density stratification, this expression is no longer valid. The fluid density is a function of depth. As the particle falls, the density of the surrounding fluid keeps changing. Therefore, the particle velocity is a function of density, depth or time. In fact, no finite terminal velocity can be reached in a linealy density stratified fluid of infinite extent. This expression is used here for quasi state terminal fall velocity based on the local ambient fluid density.

For density ratios greater than 0.625, the coefficients a and b in equation (2-8) become imaginary. When the density ratio is equal to 0.625 , the solution is special because the coefficients a and b are equal. The solution is found as

$$V=\frac{(\frac{\rho_s}{\rho}-1)gd^2}{18\nu}\left[1-\frac{2a'\sqrt{t}}{\sqrt{\pi}}+(2a'^2t-1)e^{a'^2t}\,\mathrm{erfc}(a'\sqrt{t})\right] \qquad (2\text{-}12)$$

in which

$$a' = a = b = \frac{4\sqrt{\nu}}{d} \qquad (2\text{-}13)$$

II-5 Solution for Particle Velocity above the Stokes Range

The solution to equation (2-2) can be obtained as following ing by holding K as a constant with respect to time.

(a) For $\frac{g\nu}{(\frac{\rho_s}{\rho}+\frac{1}{2})^2} \geq \frac{8\nu K}{(\frac{\rho_s}{\rho}+\frac{1}{2})}$,

$$V = \left[\frac{(\frac{\rho_s}{\rho}-1)g}{\frac{\rho_s}{\rho}+\frac{1}{2}}\right]\left[\frac{1}{a_1 b_1} + \frac{e^{a_1^2 t}}{a_1(a_1-b_1)}\,\mathrm{erfc}(a_1\sqrt{t}) - \frac{e^{b_1^2 t}}{b_1(a_1-b_1)}\,\mathrm{erfc}(b_1\sqrt{t})\right] \qquad (2\text{-}14)$$

in which

$$a_1, b_1 = \frac{9\sqrt{\nu}}{2(\frac{\rho_s}{\rho}+\frac{1}{2})d} \pm \frac{3}{2}\sqrt{\frac{g\nu}{(\frac{\rho_s}{\rho}+\frac{1}{2})^2} - \frac{8\nu K}{(\frac{\rho_s}{\rho}+\frac{1}{2})}} \qquad (2\text{-}15)$$

(b) For $\frac{g\nu}{(\frac{\rho_s}{\rho}+\frac{1}{2})^2} = \frac{8\nu K}{(\frac{\rho_s}{\rho}+\frac{1}{2})}$

$$V = \left[\frac{(\frac{\rho_s}{\rho}-1)g}{(\frac{\rho_s}{\rho}+\frac{1}{2})}\right]\left[1 - \frac{2a_1'\sqrt{t}}{\sqrt{\pi}} + (2a_1'^2 t - 1)e^{a_1'^2 t}\,\mathrm{erfc}(a_1'\sqrt{t})\right] \qquad (2\text{-}16)$$

in which

$$a_1 = b_1 = a_1' = \frac{4\sqrt{\nu}K}{d} \qquad (2\text{-}17)$$

(c) The steady state solution is

$$V = V_T = \frac{(\frac{\rho_s}{\rho}-1)gd^2}{18\nu K} = f(\mathbb{R}) \qquad (2\text{-}18)$$

II-6 Numerical Computation of the Particle Displacement

In general, the particle displacement fields are calculated by integration directly from the velocity fields by means by computer. However, it is still very interesting to show the analytical form for the displacement of the sphere resulting from the intergration of the velocity field. The displacement —— time relation can be compared directly with the experimental results to verify the proposed mathematical model, equations (2-1) and (2-2). Taking the motion of the sphere within stokes range as an example, the displacements Z are =

(a) For $\frac{\rho_s}{\rho} \gtrless 0.625$

$$Z = V_T\left\{t + \frac{b}{a^2(a-b)}\left[e^{a^2t}\operatorname{erfc}(a\sqrt{t}) - 1 + \frac{2a\sqrt{t}}{\sqrt{\pi}}\right] - \frac{a}{b^2(a-b)}\left[e^{b^2t}\operatorname{erfc}(b\sqrt{t}) - 1 + \frac{2b\sqrt{t}}{\sqrt{\pi}}\right]\right\} \quad (2\text{-}19)$$

(b) $\frac{\rho_s}{\rho} = 0.625$

$$Z = V_T\left\{t + \frac{3}{16} - \frac{3\sqrt{t}}{2\sqrt{\pi}} + \left(2t - \frac{3}{16}\right)e^{16t}\operatorname{erfc}(4\sqrt{t})\right\} \quad (2\text{-}20)$$

In the present study, the displacement field is obtained from the integration of the velocity field. The solution which have been obtained for particle velocity, within and above the stokes range, are based upon K and ρ being held constant locally in time.

As in fact, both the fluid density and the correction coefficient are variables. Consequently, these solutions are approximate in that these two parameters are determined using local values averaged over the time increments. The procedures for the numerical computation of the velocities are (5):

a- For the first time step, compute the constants a and b and then V for a given fluid viscosity, sphere diameter, the sphere to ambient fluid density ratio and the time increment.

b- Using the results of step (a) compute K and ρ for the next time increment.

c- Repeat step (b) by using the results from step (b) for new K, ρ and time increment and so on. Where velocities over two successive time intervals are averaged for each computation followed.

The error function is computed directly using IBM 360 computer in which the solution, equation (2-9) is computed using complex arguments. The real part of V is the desired quantity since it can be shown that V is a real number. The complex error function is evaluated using the infinite series approximation as shown below. The convergence of the series is rather fast.

$$\begin{aligned} erfc(x+iy) &= 1 - erf(x+iy) \\ &= 1 - \Big\{ erf(x) - \frac{e^{-x^2}}{2\pi x}\left[(1-\cos 2xy) + i\sin 2xy\right] \\ &\quad + \frac{2}{\pi} e^{-x^2} \sum_{n=1}^{\infty} \frac{e^{-(\frac{n^2}{4})}}{n^2+4x^2}\left[f_n(x,y) + i g_n(x,y)\right] + \epsilon(x,y) \Big\} \end{aligned} \qquad (2\text{-}21)$$

in which

$$f_n(x,y) = 2x - 2x(\cosh ny)(\cos 2xy) + n(\sinh ny)(\sin 2xy) \qquad (2\text{-}22)$$

$$g_n(x,y) = 2x(\cosh ny)(\sin 2xy) + n(\sinh ny)(\cos 2xy) \qquad (2\text{-}23)$$

$$\epsilon(x,y) = 10^{-16}\left|erf(x+iy)\right| \qquad (2\text{-}24)$$

The real error function is computed by using the rational approximation for $|x| \leq 3$ and asymtotic expansion for $|x| > 3$.

(a) For $|x| \leq 3$.

$$erf(x) = 1 - (1 + a_1 x + a_2 x^2 + \cdots + a_6 x^6)^{-16} + \epsilon(x) \qquad (2\text{-}25)$$

in which

$$\begin{aligned} A_1 &= 0.0705230784 \\ A_2 &= 0.0422820123 \\ A_3 &= 0.0092705272 \\ A_4 &= 0.0010520143 \\ A_5 &= 0.0002765762 \\ A_6 &= 0.0000430638 \end{aligned} \qquad (2\text{-}26)$$

and

$$|\epsilon(x)| \leq 3\times 10^{-7} \qquad (2\text{-}27)$$

(b) For $|x| > 3$

$$\mathrm{erf}(x) = 1 - \frac{e^{-x^2}}{\sqrt{\pi}\,x}\left[1 + \sum_{n=1}^{\infty}(-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2x^2)^n}\right] \qquad (2\text{-}28)$$

CHAPTER III- EXPERIMENTAL CONSIDERATIONS

III-1 Experimental Apparatus

The motion of the free falling sphere was performed inside a test tank where density stratified fluid was filled. The tank was cylindrical with inside diameter of 1 ft and height of 6 ft. This tank was made of transparent lucite so that pictures of particle motion could be taken by means of photographic techniques. Figures (1) and (2) show the set up photographically and schematically. As is shown in figures, scale with minimum readings of 1/16 inches was installed to measure the particle displacement. In addition, the time scale associated with the particle displacement was measured by a digital time counter. A universal counter-timer, Model 901 from Dynasciences Corporation was used (See Figure 3). The timer enables the time recording of accuracy to 0.001 second.

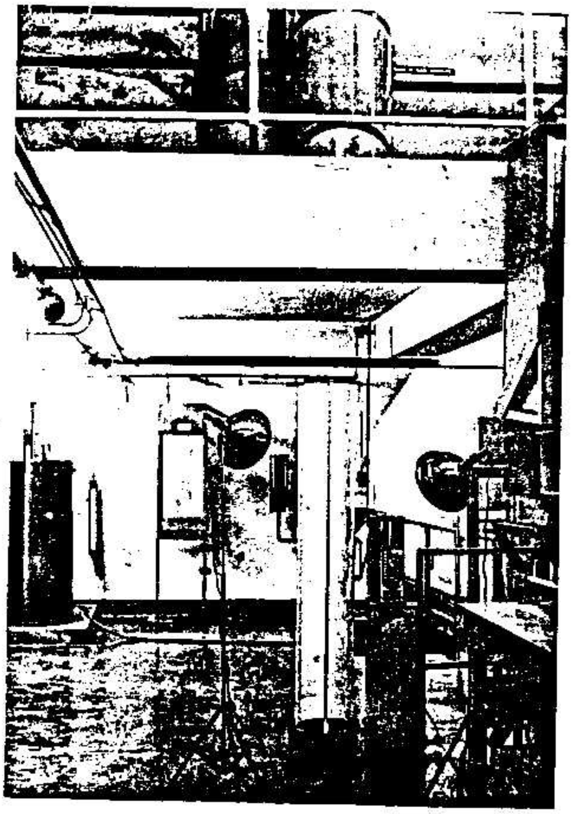

FIGURe 1- GENERAL VIEW OF THE EXPERIMENTAL APPARATUS

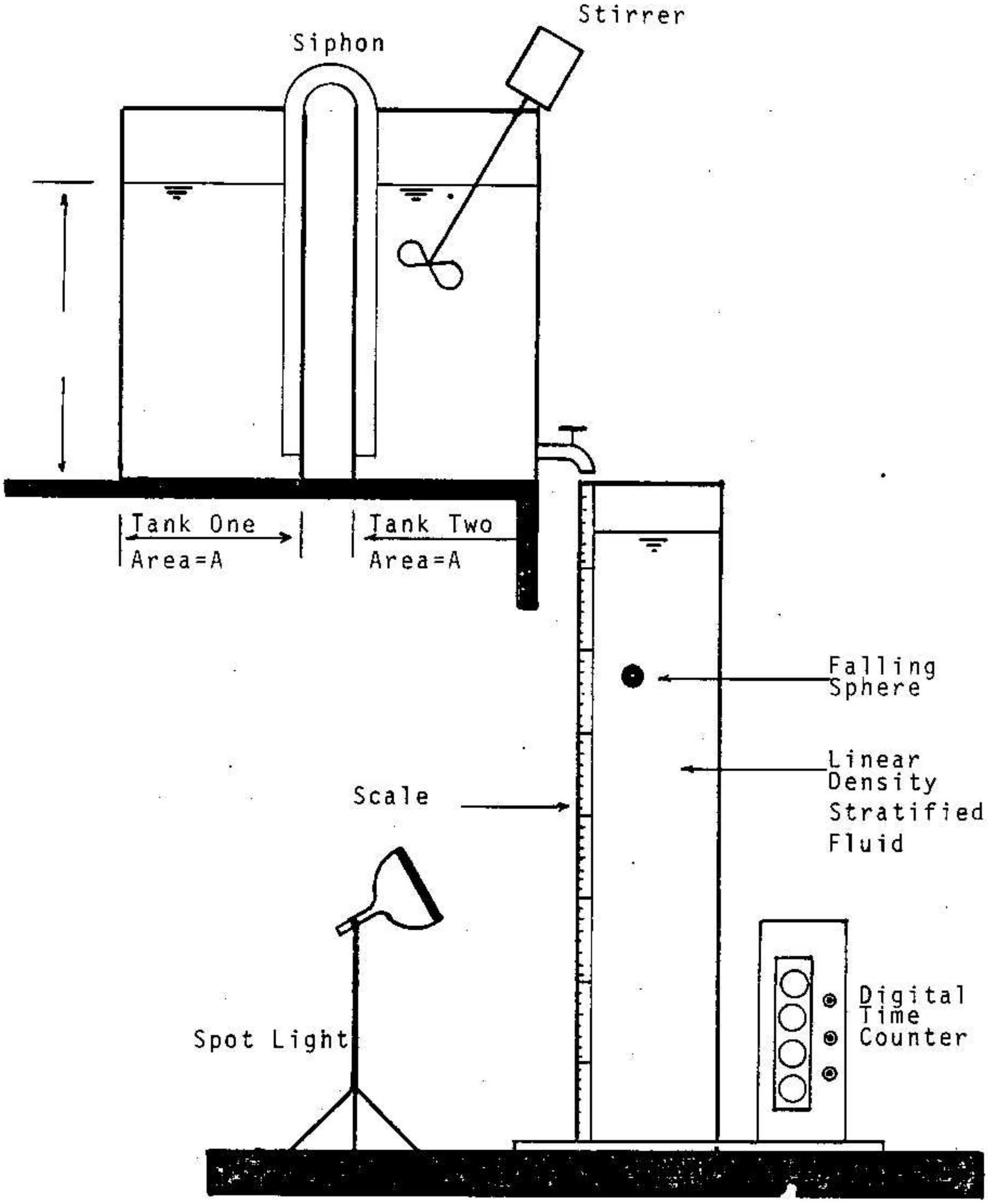

FIGURE 2- Schematic Diagram of the Experimental Set-Up

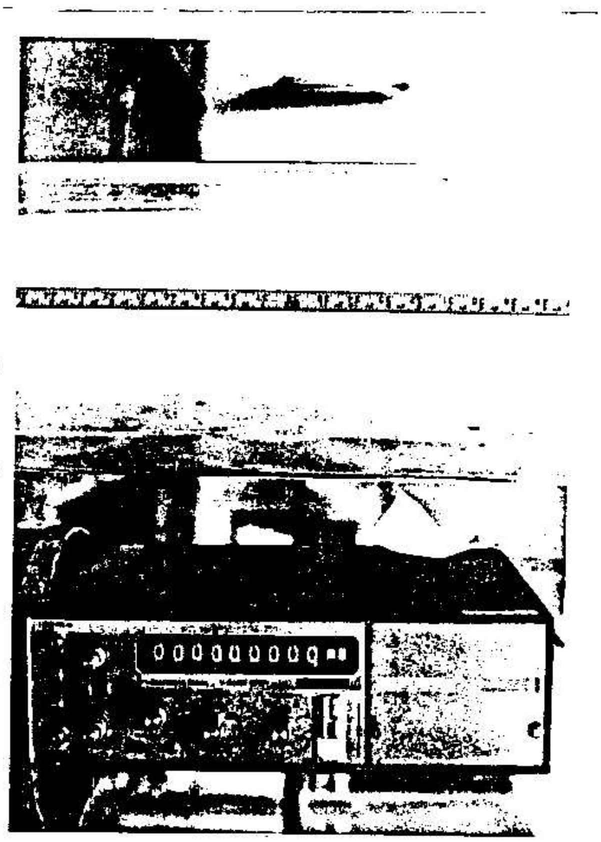

FIGURE 3- MEASUREMENTS OF TIME AND PARTICLE DISPLACEMENT

The pictures of the particle motion, fixed scale, and digital timer were taken by a camera about 40 feet away from the test section. Advantage of this scheme of taking data was the minimization of the inherent error so that data correction could be neglected. Sources of errors were photographic due to the distortion of the scale and images of the sphere through use of the telephoto lens. A Nikon F camera with Nikon F-36 motor driver and Nikkor-Auto 300^{mm} F-4.5 lens was used. The motor unit enabled sequential photographys to be taken at a rate of approximately four exposures per second. Kodak Tri-X Pan film with ASA 400 was used. With two reflector spot lights totaling 300 watts, the best exposure settings were f/4.5 at 1/1000 second. This provided a short duration, high contrast picture of the black sphere against the white background. In addition, this shutter speed enabled time records to be made by the incorporation of digital time counter. The films of the particle motion were enlarged for the ease of reading by means of the enlarger. Data of the particle displacement and time were takenfor the comparison with the analytical results to verify the mathematical model.

Black hollow plastic beeds with tap water inside were used as the particles for this experiment. The diameters are 5/8 inches and their specific gravity are 1.17. This specific gravity of the sphere is slightly greater than that of the fluid. The advantage was that the sphere would not fall down to the bottom of the tank in a very short time period and enabled more film exposures being made for the entire history of the particle motion.

III-2 Creation of Density Stratified Fluid

The stratification used in this experiment was established by creating gradient of salt concentration in the test tank. The salt used for this purpose was common food grade sodium chloride, NaCl.

Following the similar stratification procedures performed by Fox (3), the apparatus used is shown schematically in Figure 2. Two tanks of identical size with 2 feet diameter and 3 1/2 feet high were connected hydraulically by a 1 inch tubing which served as a siphon (see Figure 4). Tank number one was filled with tap water. Tank number two was filled with salt water of known concentration and of the same volume as tank number one. As the salt water was continuously withdrawn into the test tank through a special filling device, the fresh water in tank number one will continuously go to salt water tank and keep the water elevation the same in two tanks via siphon. A mixer was installed at the top of salt

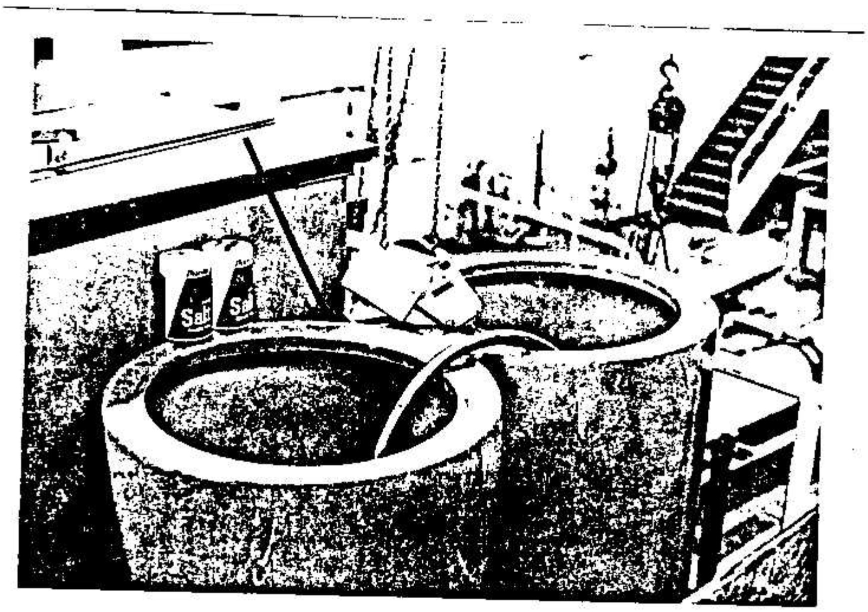

FIGURE 4- APPARATUS FOR CREATING DENSITY STRATIFIED FLUID

water tank to keep the salt water well mixed all the time. Thus, the salt concentration decreased as the time increased due to the continueous withdraw of salt water. Finally, the effluent from salt water tank being accumulated in the test tank exhibitted a linear variation in density with elevation. This method was to carefully float lighter fluid atop heavier fluid. A special filling device was required to reduce the vertical mixing. The filling mechanism was composed of a floating plate on which a vertical intake tubing was attached. This tubing was connected to the discharge valve of the salt water tank. Keeping the flow rate of the effluent as small as possible, one would be able to convert the vertical momentum of the fluid entering the device to a very slight horizontal momentum so the fluid gently spreaded out on the surface. This type of creating laboratory stratifications is rather time consuming. For instance, it took about ten hours to complete the filling process. Another type of filling mechanism is to lift up lighter fluid by underriding it with heavier fluid. In that case the tank number one should be filled with salt water and with fresh water in tank number two. An inverted funnel placing at the bottom of the tank can be used as the filling device. The latter method was used for this experiment since the filling time is shorter than the former method.

In the stratification procedures adopted, it can be proved mathematically that the variation in fluid density (or salt concentration) with depth is indeed linear. Referring to Figure 2, tank one and two have identical cross-sectional areas A. The depth of water in both tank is H above the tank base. The initial conditions are: t = 0, H = Ho, the initial height; C = 0 in tank two; C = Co in tank one. C is the salt concentration at any time in % by weight. Applying one-dimensional mass balance to the system, the following relation is obtained.

$$\frac{d\forall}{dt} + Q = 0 \qquad (3\text{-}1)$$

where ∀ is the total volume of fluid in two tanks. Q is the discharge from tank two and is proportional to $H^{1/2}$ from energy consideration. Hence

$$2A\frac{dH}{dt} = -k\sqrt{H} \qquad (3\text{-}2)$$

where k is a proportional constant. Consideration of the conservation of mass for the concentration species yields,

$$\frac{d(\forall c)}{dt} + QC = 0 \quad . \tag{3-3}$$

or

$$\frac{d(HAC_o)}{dt} + \frac{d(HAC)}{dt} + kC\sqrt{H} = 0 \tag{3-4}$$

Substituting equation (3-2) into (3-4), one arrives

$$\frac{dH}{H} = \frac{dc}{C-C_o} \tag{3-5}$$

The solution to this equation becomes

$$\ln(C-C_o)\Big|_o^C = \ln H\Big|_{H_o}^H \tag{3-6}$$

After evalution of the limits of integration, the final solution is,

$$C = C_o - \frac{C_o}{H_o} H \cdot \qquad (3\text{-}7)$$

Thus, the result shows the salt concentration of the effluent is a linear function of H. Consequently, the fluid density decreases linearly in the test tank as the depth of fluid increases. The relation between ρ and C is as following

$$\rho = \rho_w (1 + C) \qquad (3\text{-}8)$$

ρ_w is the density of the fresh water.

III-3 Measurement of Density Profile

In order to verify experimentally that the density stratification is indeed linear, density field was measured. The density gradient was calculated from the data of density profile in the test tank. Fluid density was measured indirectly by means of a conductivity probe in conjunction with a conductivity bridge. The fluid density was correlated with and represented by the conductivity of the salt solution. A YSI Model 31 Conductivity Bridge and YSI Model 3402 Conductivity Cell as shown in Figure 5 were used for this measurement. This required the calibration work. Several standard salt solutions of different known concentrations were prepared. The unit for the concentration of the solution was in % by weight. The unit for the output from the conductivity bridge was in micromhos. The calibration curve is shown in Figure 6.

As soon as the stratification procedures were finished, measurements of the density were made by traveling the probe vertically downwards and making several stops. The readings from the conductivity bridge were taken at each stop and the corresponding concentrations were found. A typical density profile measured is shown in Figure 7. The result shows that the variation in density with depth of water is linear The density gradient is equal to the slope of the straight line.

III-4 General Experimental Steps

The following steps were performed in each run,

1. Fill up the test tank with density stratified fluid.
2. Measure the density profile and calculate the density gradient.

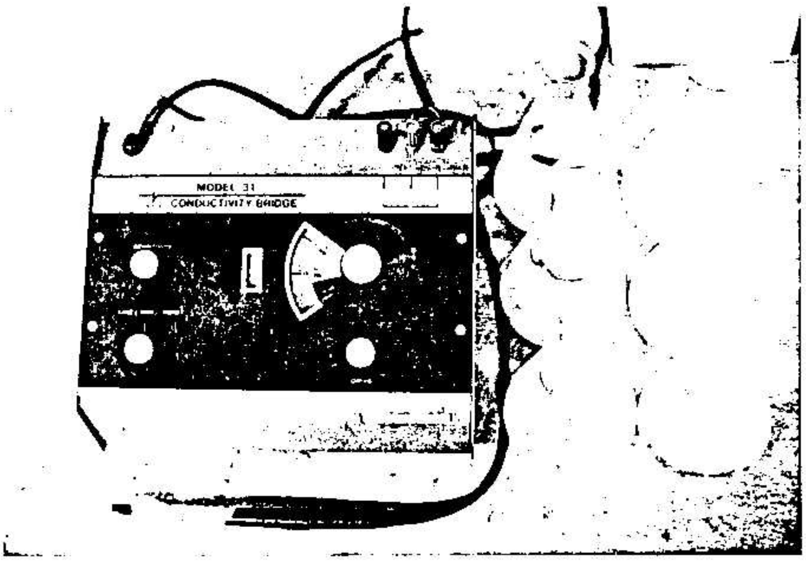

FIGURE 5- CONDUCTIVITY BRIDGE AND CELL FOR DENSITY MEASUREMENT

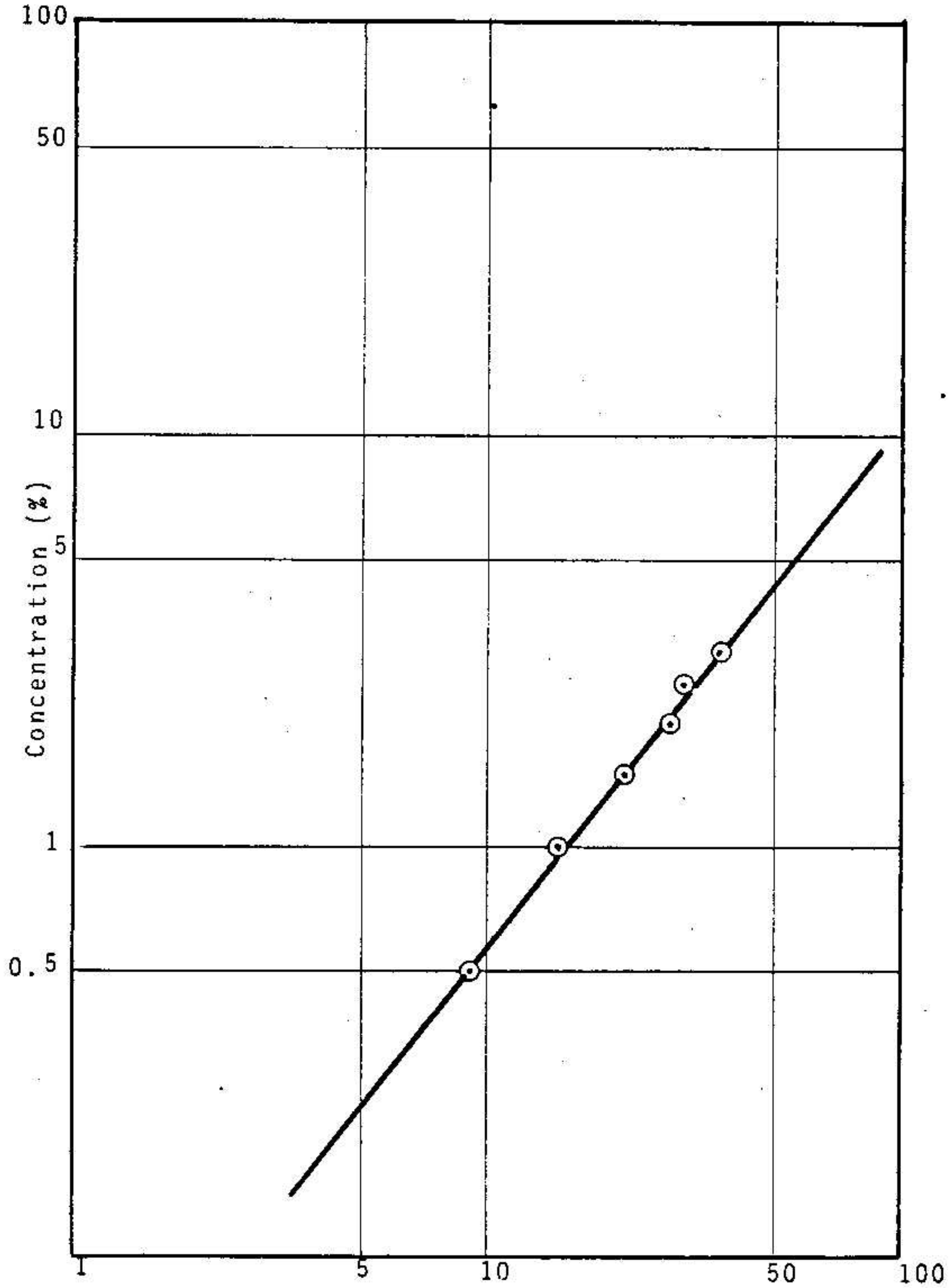

FIGURE 6- Calibration Curve for the Measurement of Density (Micromhos)

3. Turn on the digital time counter.
4. Release the sphere and take the picture of particle motion.
5. Develope the film.
6. Take the data of particle displacement and time from the film through the use of enlarger.
7. Plot dimensionless quantities of particle displacement against time and compare with analytical results.

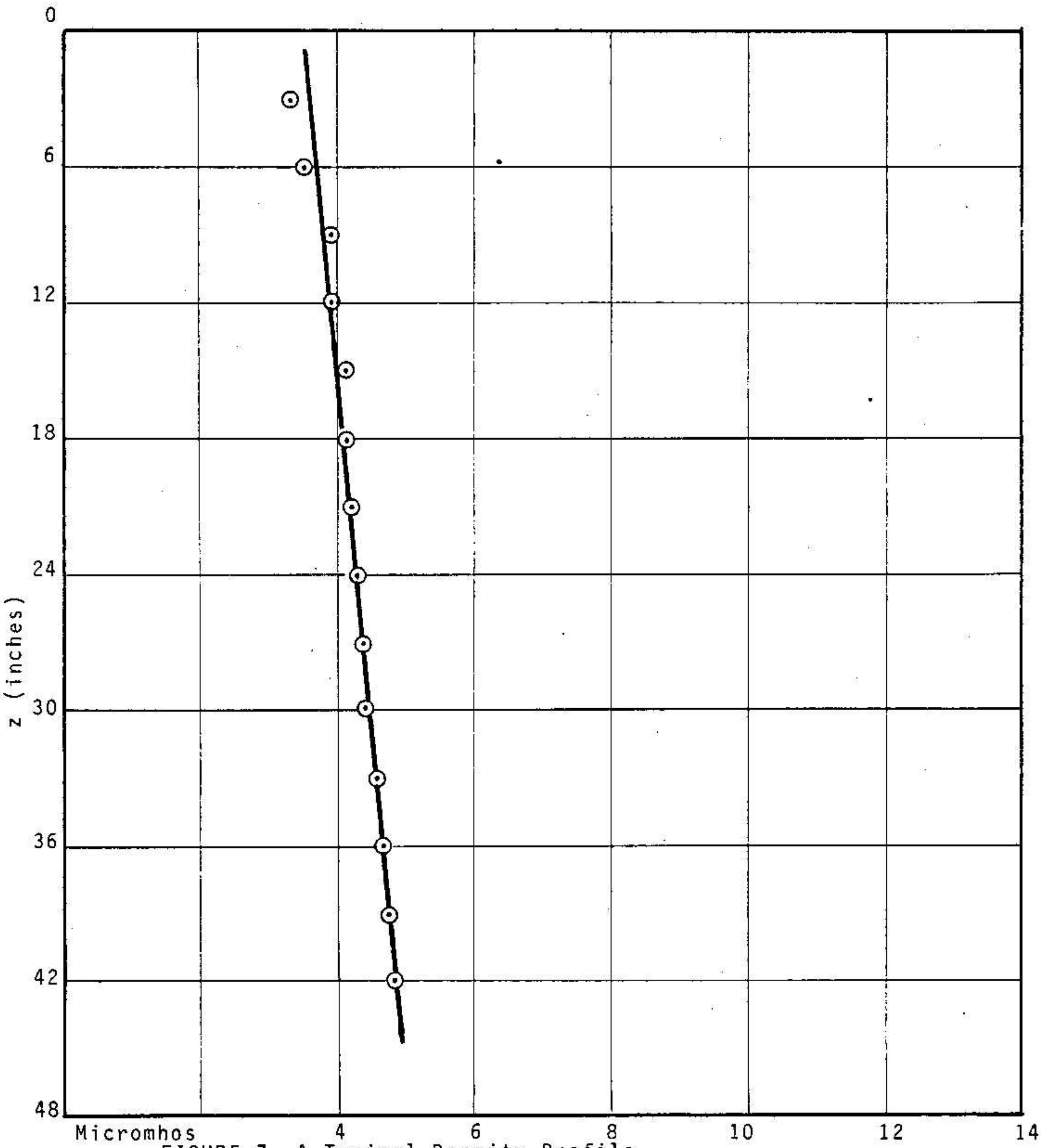

FIGURE 7- A Typical Density Profile

CHAPTER IV- RESULTS AND DISCUSSIONS

Particle motion in several different density gradients of the linear density stratified fluid have been investigated by means of studying the mathematical model. The time history of the particle displacement is of particular interesting. The dimensionless displacement versus dimensionless time are plotted in figure 8 for different fluid density gradients. They are numerical results from the computer. As can be seen from the curves, the displacement of particle is indeed a function of the fluid density gradient. The motion of the particle is slower with the increase of the density gradient. To get an idea of the order of magnitude of the particle motion, the total displacements of the free falling particle (starting at zero velocity) at the end of the first second in different density gradients are tabulated in Table I. At that moment, the particle has already reached its terminal fall velocity based on the local ambient fluid density.

The results shown above are based on the steel particle with specific gravity of 7.8. It is noted that the differences in particle displacements for various density gradients are not very appreciable. This can be seen clearly in Figure 8 that the difference in displacement between density gradients of 0.005 $slug/ft^4$

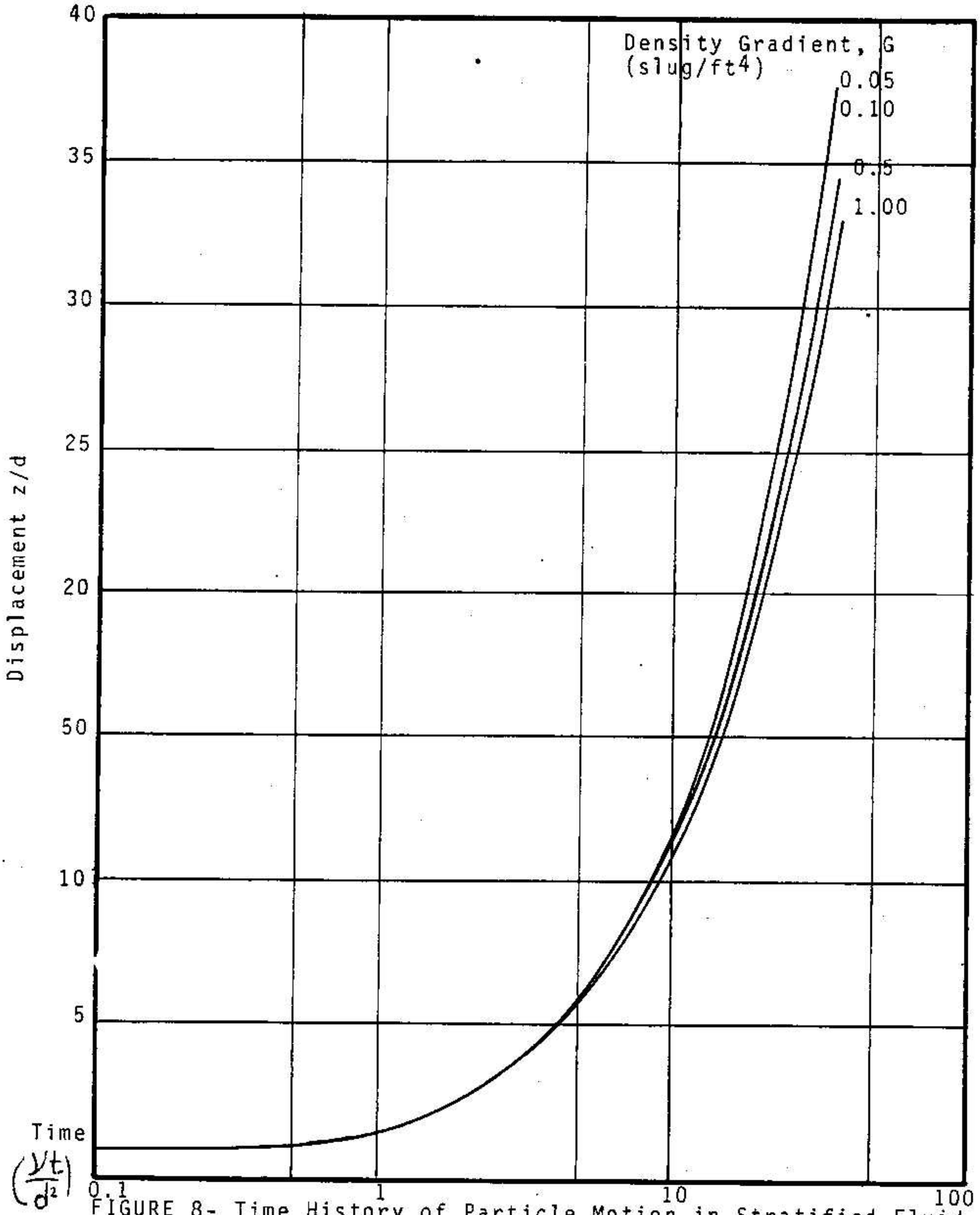

FIGURE 8- Time History of Particle Motion in Stratified Fluid of Different Density Gradients (ρ_s=15.09 slug/ft^3)

TABLE 1

Particle Motion in Different Density Gradiente

Density Gradient G $\left(\frac{slug}{ft^4}\right)$	*Displacement $\frac{Z}{d}$	$\frac{V}{V_T}$
0.005	35.9425	1
0.030	35.7572	1
0.050	35.6114	1
0.100	35.2563	1
0.250	34.1510	1
0.500	32.7094	1
0.750	31.4628	1
1.000	30.4543	1

* at t = 1 second or $\frac{\nu t}{d^2}$ = 32.5949

notations:

G = Density Gradient of Fluid

Z = Particle Displacement

d = Particle Diameter

V = Particle Velocity

V_T = Terminal Fall Velocity Based on Density of Fluid at Z

t = Time

ν = Kinematic Viscosity of Fluid

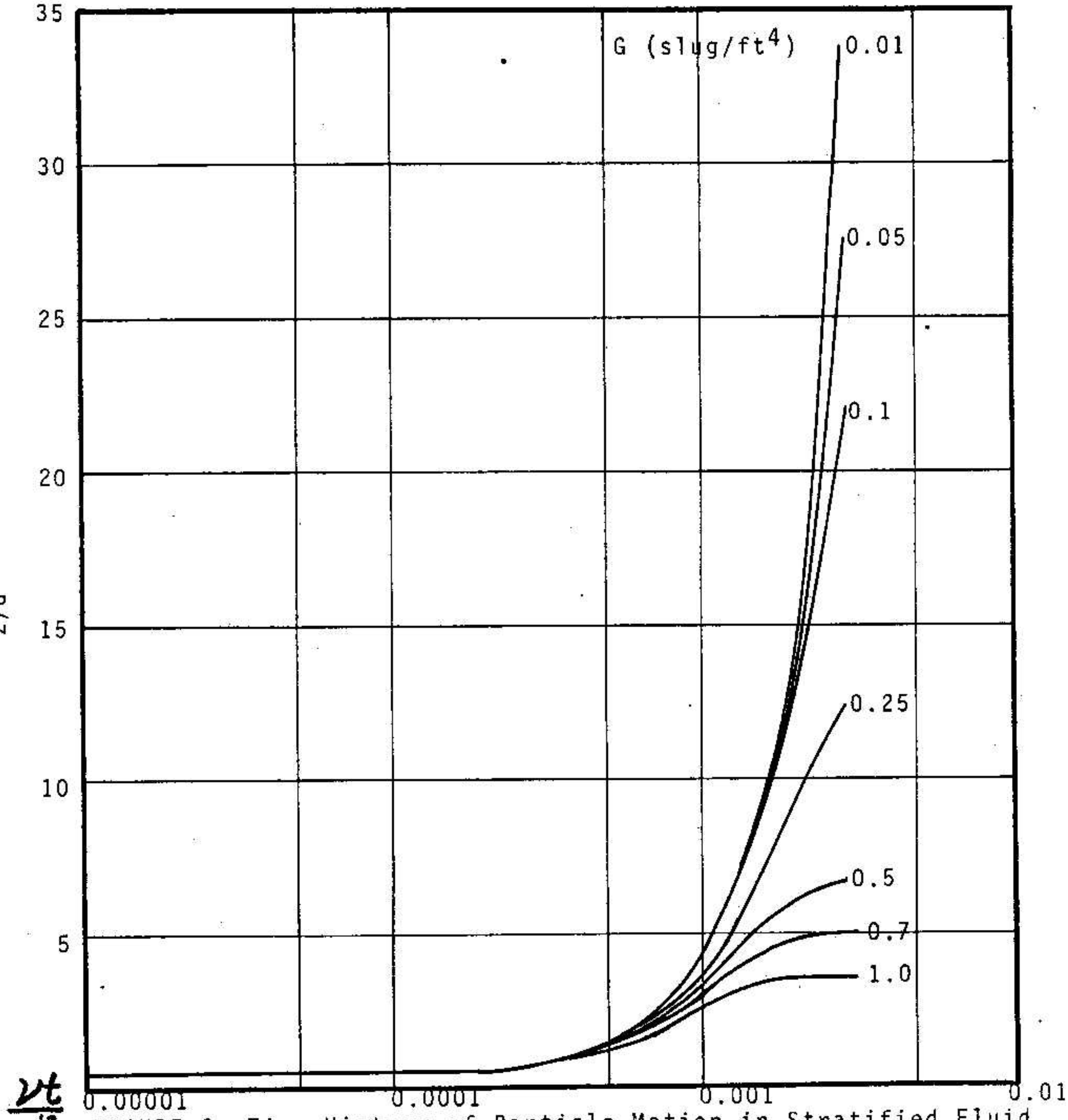

FIGURE 9- Time History of Particle Motion in Stratified Fluid of Different Density Gradients (ρs=2.25 slug/ft^3)

and 1.0 slug/ft^4 is about five particle diameters at the end of the first second. For the case of particle specific gravity of 1.17, the results are in contrast to the previous case. Figure 9 shows the time history of the particle motion with different density gradients for particle specific gravity of 1.17. The difference in displacements between density gradients of 0.01 slug/ft^4 and 1.0 slug/ft^4 is significant. At low density gradients, the particle motions behave similar to the previous case. However, at high density gradients, the diversion is big. The density of the fluid increases rapidly as the depth increases. The densities of fluid at deeper levels are almost equal to the particle density. Therefore, the solid particle behaves like a neutral buoyant particle. The total displacement tends to remain to be constant. In other words, the particle starts at zero velocity at the free surface and accelerates a little bit to gain some velocity. Then, the particle tends to slow down and finally stay at certain elevation. At that level, the particle velocity is about equal to zero and the particle density is equal to the ambient fluid density. This is indicated by the lowest curve in Figure 9. The density gradient for this curve is unity. The curve approaches to horizontal line as time goes to infinite. The terminal position of the particle can be determined from the figure.

The same tendency is also found for other curves with density gradients of 0.25 slug/ft^4 and 0.5 slug/ft^4 since the fluid media under consideration is an infinitive body and the linear density gradient is postulated without any boundary condition at the bottom of the fluid. In natural water body with thermocline, the thickness of the thermocline is finite. The fluid density at the bottom of this layer is equal to the uniform density in the hypolimnion. Hence, the phenomena just described are not very common except that the particle may stay inside the thermocline if its density is almost equal to the ambient fluid density.

Typical experimental results are shown in Figure 10, 11, 12 for density gradients of 0, 0.036 slug/ft^4 and 0.072 slug/ft^4 respectively. The dots are experimental data. The solid line is the theoretical result. The case of zero density gradient means the homogeneous fluid. It is presented here for the purpose of comparison with other cases having the density gradients. It can be seen from these figures that the agreements between the theoretical analysis and the experimental data are quite good. A few scattering of data is shown at the beginning the motion. This is probably due to the side swing of the particle as it falls down. The small latteral movement is caused by

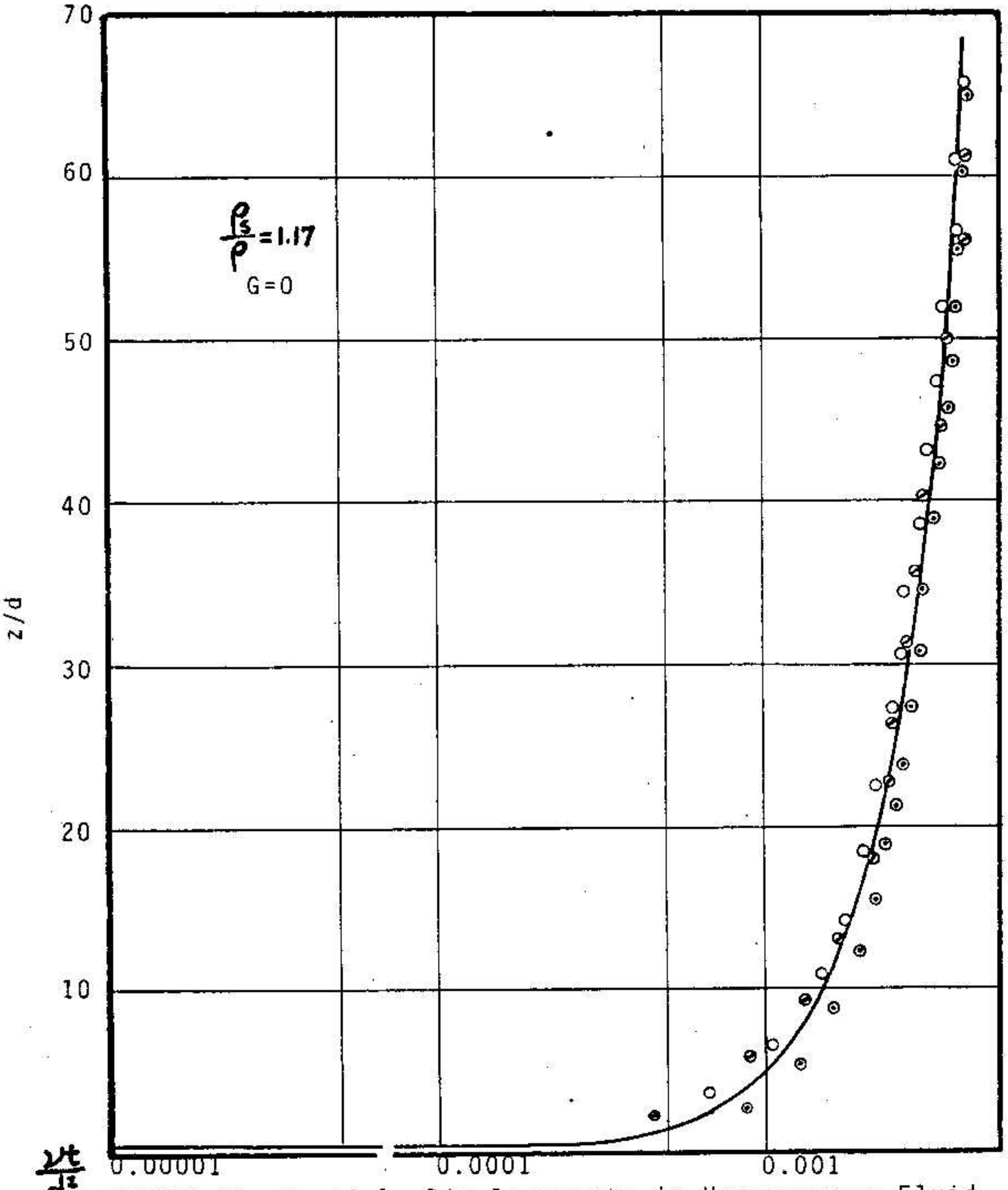

FIGURE 10- Particle Displacements in Homogeneous Fluid

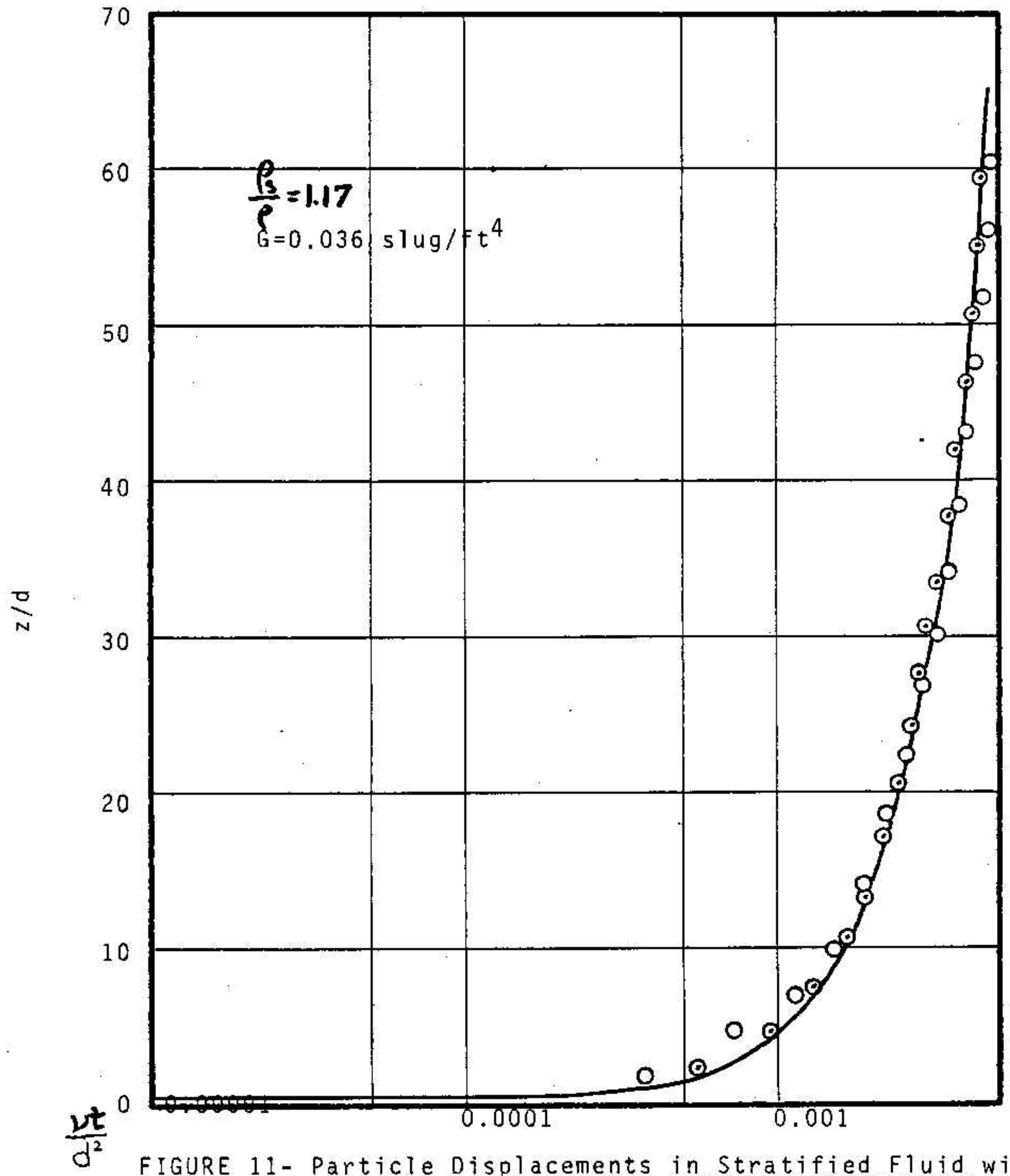

FIGURE 11- Particle Displacements in Stratified Fluid with Density Gradient G=0.036 slug/ft^4

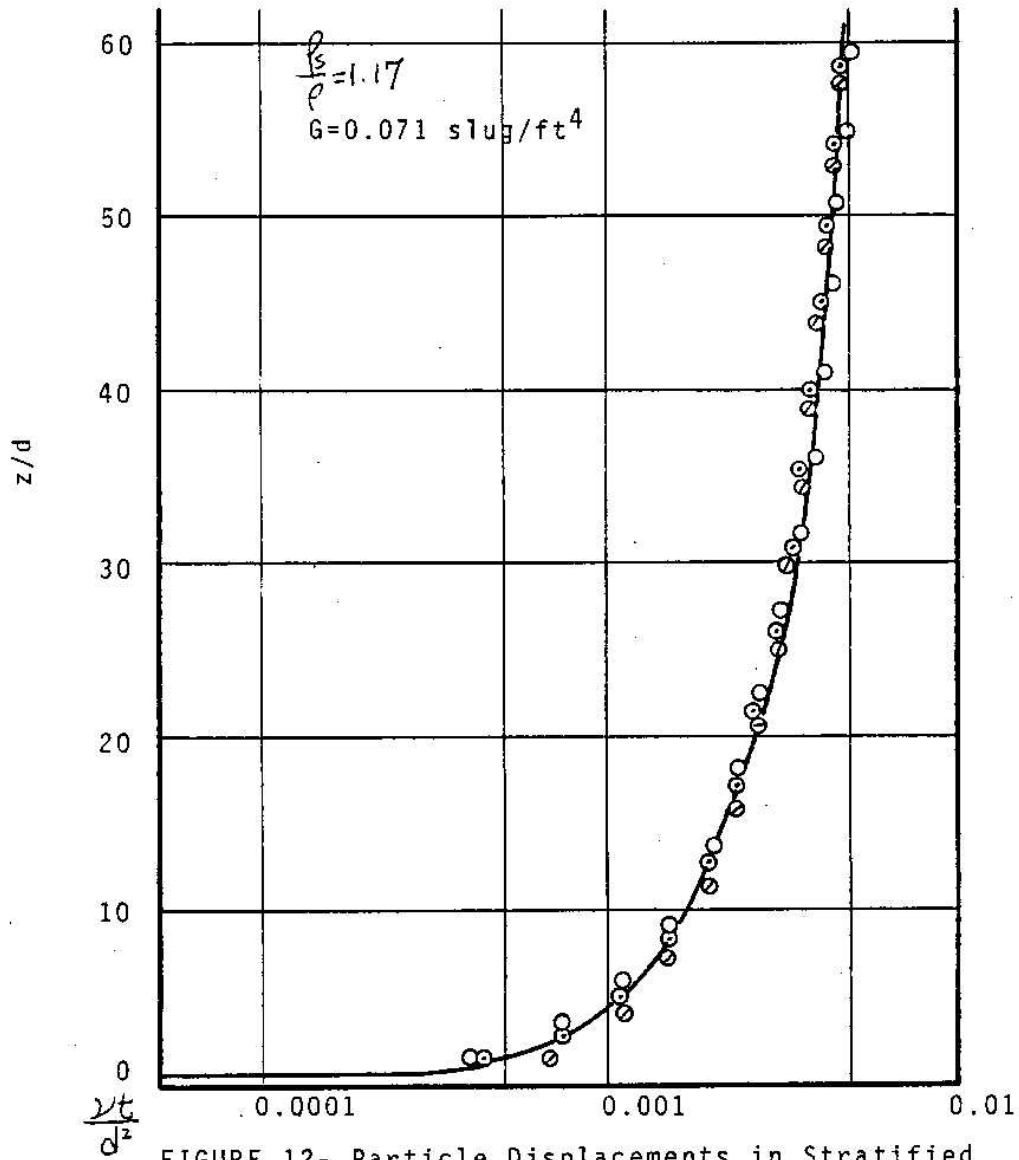

FIGURE 12- Particle Displacements in Stratified Fluid with Density Gradient G=0.071 slug/ft^4

the rotation of the particle. In general, extremly careful releasing of the particle must be made in order to keep the straight downward movement of the particle. Particle motions in an environment of high density gradients were not investigated experimentally due to the great amount of salt involved . To create high density gradients in the laboratory is not very realistic as far as the simulation of density stratification in natural water body is concerned. However, the results from the theoretical analysis will be able to predict the particle motion in such a condition.

CHAPTER V- CONCLUSIONS

As the result of this study, the following conclusions are made.

1- The equations governing the particle motion in a viscous fluid of homogeneous density can be applied to the density stratified fluid if the density stratified fluid body is considered to be composed of infinite homogeneous layers of finite thickness. The results of theoretical analysis have good agreements with the experimental data.

2- The particle motion due to gravity in a linear density stratified fluid is a function of fluid density gradient. For a particle of given density and diameter, the motion is getting slower with the increase of the fluid density gradient if their surface fluid densities are the same. The particle velocity also decreases with the increasing fluid depth due to the fluid density increase for a given fluid density gradient beyond the zone of initial accelerated motion.

3- For high particle-fluid density ratios, the effect of fluid density gradient on the particle motion is not appreciable. For smaller ratios of greater than one, the effect is very pronounced.

4. For the particle-fluid density ratios almost equal to one, the particle will stay at a certain elevation as a neutral buoyant particle whose density is equal to the ambient fluid density. The terminal position can be determined.
5. There is no terminal fall velocity for particle motion in stratified fluid as can be defined like the case of homogeneoos fluid. However, a quasi terminal fall velocity can be defined based on local ambient fluid density.

CHAPTER VI- REFERENCES

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CHAPTER VII- APPENDIX I- NOTATIONS

The following symbols are used in the report:

A	cross-sectional area of tank
a, a_1 a', a_1'	coefficients, see Eqs. (2-8), (2-13), (2-15), (2-17)
b, b_1	coefficients, see Eqs. (2-8), (2-15)
C_D	drag coefficient
c	salt concentration (Co for initial concentration)
d	particle diameter
e	base of natural logarithms
erf	error function
erfc	complementary error function
fn	function, see Eq. (2-22)
G	density gradient
g	acceleration due to gravity
gn	function, see Eq. (2-23)
H	water depth in tank
h	depth measured from the top of the thermocline
i	imaginary unit $\sqrt{-1}$ or an integer in a series
K	correction coefficient of the Stokes drag
k	proportional constant
L {V(t)}	Laplace transformation of V(t)
m	mass of fluid displaced by the particle
ms	particle mass
n	an integer in a series
Q	discharge of salt water

$\mathbb{R}$	particle Reynolds number
S	parameter used in the Laplace transform
t	time
$\forall$	volume
V	particle velocity
V_T	particle terminal fall velocity
X	real part of the complex number
Y	total depth of the test tank or the thickness of the thermocline
y	imaginary part of the complex number, also a dummy variable
z	particle displacement measured from the free surface
ρ_s	density of the particle
ρ_w	density of fresh water
	density of fluid (ρ_o for density at the bottom of the test tank)
E	error
μ	dynamic viscosity of the fluid
ν	kinematic viscosity of the fluid
τ	dummy variable

<u>APPENDIX II- COMPUTER PROGRAM</u>

```
C   ACCELERATED MOTION OF A PARTICLE IN A VISCOUS FLUID OF CHANGING DENS
      DIMENSION T(1000),Y(1000)
      COMPLEX CMPLX,CEXP,CSQRT
      COMPLEX A,B,ZA,ZB,SUMAN,SUMBN,SUMA,SUMB,ARGA2,ARGB2,ZCA,ZCB
      COMPLEX ARGA1,ARGB1,ZXYA,ERFCA,ZXYB,ERFCB,VS
      DELT=0.001
      D=1.0/96.0
      RHOS=7.8*1.935
      RHO0=1.935
      H=5.0
      VIS=0.00354
      G=32.1725
      PI=3.14159
      VSRI=0.0
      DO 15 I=1,1000
      Y(I)=0.0000
15    T(I)=0.0001
      Y(1)=D/2.
      T(1)=0.0001
      DO 45 I=1,1000
      RHO=RHO0+1.000*(Y(I)-(D/2.))*1.935
      VT=((RHOS/RHO)-1.)*G*(D**2)/(18.*VIS)
      P1=(T(I)*VIS)/D**2
      P2=(Y(I)/D)
      C=9.*SQRT(VIS)/(2.*((RHOS/RHO)+(1./2.))*D)
      H1=9.*VIS/(((RHOS/RHO)+(1./2.))**2)
      H2=3.*VIS/((RHOS/RHO)+(1./2.))
      IF(H1-H2)11,12,12
11    A=C+(1.5/D)*CMPLX(0.,SQRT(H2-H1))
      B=C-(1.5/D)*CMPLX(0.,SQRT(H2-H1))
      XA=REAL(A*SQRT(T(I)))
      YA=AIMAG(A*SQRT(T(I)))
      XB=REAL(B*SQRT(T(I)))
      YB=AIMAG(B*SQRT(T(I)))
      SUMA=CMPLX(0.,0.)
      SUMB=CMPLX(0.,0.)
      DO 35 N=1,10
      RN=N
      FFA=2.*XA-2.*XA*COSH(RN*YA)*COS(2.*XA*YA)+RN*SINH(RN*YA)*SIN(2.*X
     1*YA)
      FGA=2.*XA*COSH(RN*YA)*SIN(2.*XA*YA)+RN*SINH(RN*YA)*COS(2.*XA*YA)
      FFB=2.*XB-2.*XB*COSH(RN*YB)*COS(2.*XB*YB)+RN*SINH(RN*YB)*SIN(2.*X
     2*YB)
      FGB=2.*XB*COSH(RN*YB)*SIN(2.*XB*YB)+RN*SINH(RN*YB)*COS(2.*XB*YB)
      ZA=CMPLX(FFA,FGA)
      ZB=CMPLX(FFB,FGB)
      SUMAN=((EXP(-(RN**2)/4.))/(RN**2+(4.*(XA**2))))*ZA
      SUMBN=((EXP(-(RN**2)/4.))/(RN**2+(4.*(XB**2))))*ZB
      SUMA=SUMAN+SUMA
35    SUMB=SUMBN+SUMB
      ARGA2=(2./PI)*EXP(-XA**2)*SUMA
      ARGB2=(2./PI)*EXP(-XB**2)*SUMB
      SA=1.-COS(2.*XA*YA)
      DA=SIN(2.*XA*YA)
      ZCA=CMPLX(SA,DA)
      SB=1.-COS(2.*XB*YB)
      DB=SIN(2.*XB*YB)
      ZCB=CMPLX(SB,DB)
```

```
          ARGA1=((EXP(-XA**2))/(2.*PI*XA))*ZCA
          ARGB1=((EXP(-XB**2))/(2.*PI*XB))*ZCB
          V1=0.0705230784
          V2=0.0422820123
          V3=0.0092705272
          V4=0.0001520143
          V5=0.0002765672
          V6=0.0000430638
          EPSI=3E-7
          IF(ABS(XA)-3.)13,13,14
   13     ERFXA=1.-(1./(1.+V1*XA+V2*(XA**2)+V3*(XA**3)+V4*(XA**4)+V5*(XA**5
         3+V6*(XA**6))))+EPSI
          GO TO 18
   14     RMA=1.
          SUMAA1=-1./(2.*(XA**2))
          SUMAA2=-1./(2.*(XA**2))
   16     RMA=RMA+1.
          SUMAA1=SUMAA1*((2.*RMA-1.)/(2.*(XA**2)))*(-1.)
          SUMAA2=SUMAA2+SUMAA1
          IF(RMA-10.)16,16,17
   17     ERFXA=1.-(1./(SQRT(PI)*XA*(EXP(XA**2))))*(1.+SUMAA2)
   18     ZXYA=CMPLX(XA,YA)
          ERFCA=1.-(1./(1.-(E-16)))*(ERFXA-ARGA1+ARGA2)
          IF(ABS(XB)-3.)19,19,21
   19     ERFXB=1.-(1./(1.+V1*XB+V2*(XB**2)+V3*(XB**3)+V4*(XB**4)+V5*(XB**5)
         4+V6*(XB**6))))+EPSI
          GO TO 24
   21     RMB=1.
          SUMAB1=-1./(2.*(XB**2))
          SUMAB2=-1./(2.*(XB**2))
   22     RMB=RMB+1.
          SUMAB1=SUMAB1*((2.*RMB-1.)/(2.*(XB**2)))*(-1.)
          SUMAB2=SUMAB2+SUMAB1
          IF(RMB-10.)22,22,23
   23     ERFXB=1.-(1./(SQRT(PI)*XB*(EXP(XB**2))))*(1.+SUMAB2)
   24     ZXYB=CMPLX(XB,YB)
          ERFCB=1.-(1./(1.-(E-16)))*(ERFXB-ARGB1+ARGB2)
          VS=(((RHOS/RHO)-1.)*G/((RHOS/RHO)+1./2.))*((1./(A*B))+(1./(A*(A-B)
         5))*CEXP((A**2)*T(I))*ERFCA-(1./(B*(A-B)))*CEXP((B**2)*T(I))*ERFCB)
          VSRI=REAL(VS)
          RVSRI=VSRI/VT
          WRITE(6,40) T(I),Y(I),VT,RVSRI,P1,P2,REYNO
   40     FORMAT(7F14.4)
          GO TO 27
   12     RA=C+(1.5/D)*SQRT(H1-H2)
          RB=C-(1.5/D)*SQRT(H1-H2)
          VSR=(((RHOS/RHO)-1.)*G/((RHOS/RHO)+1./2.))*((1./(RA*RB))+(1./(RA*(
         6RA-RB)))*EXP((RA**2)*T(I))*ERFC(RA*SQRT(T(I)))-(1./(RB*(RA-RB)))*E
         7XP((RB**2)*T(I))*ERFC(RB*SQRT(T(I)))))
          RVSRI=VSR/VT
          VSRI=VSR
          WRITE(6,50) T(I),Y(I),VT,RVSRI,P1,P2,REYNO
   50     FORMAT(7F14.4)
   27     T(I+1)=T(I)+DELT
          Y(I+1)=Y(I)+VSRI*DELT
   25     IF(Y(I+1)-H)45,45,55
   45     CONTINUE
   55     CALL EXIT
```