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ANALYSIS OF A PIPE WHEN IN CONTACT WITH THE OCEAN BOTTOM AND RIGIDLY FIXED AT ONE END

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by

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TABLE OF CONTENTS

INTRODUCTION:

A long pipe of circular cross section is in contact with ground, which in this case is assumed to be a plane one. The presence of water currents give rise to hydrodynamic pressure loads along the length of the pipe and hence the deflection of the pipe. The motion of the pipe takes place quasistatically and therefore, dynamic effects are neglected. However, frictional forces resisting the motion of the pipe will be present and will depend on the nature of the ocean bottom as well as the cross section of the pipe.

Furthermore, the hydrodynamic pressure and the soil reaction may vary along the length of the pipe. In the first part of our analysis we examine the deformation of the pipe under the influence of the applied load when supported at one end and free at the other end. The system of equations governing the deformation of the rod are reduced to a non-linear integral equation, involvin. the curvature of the deflected rod. In the second part of our investigation the same pipe is pulled at the free end by a force passing through a given point. A closed solution is derived in this case in terms of elliptic integrals of the first and second kind.

BASIC EQUATIONS AND SOLUTIONS:

The equations for the bending of the pipe are derived and applied to the specific problem under consideration. The more general case, when the ocean floor is not a plane one, but a cylindrical surface is examined at a later stage. This separation has been adopted for a variety of reasons, the more important being to limit the difficulties present in the computational stage. Therefore, at first we shall concentrate our efforts in attacking the plane case.

For the plane deformation of a pipe, the equilibrium equations take the following form (Reference 1)

$$\frac{dn}{ds} + kT + q_n = 0 \qquad (1\text{-}1)$$

$$\frac{dt}{ds} - kN + q_t = 0$$

$$\frac{dm}{ds} + N = 0$$

where N, T and M are the resultant sheer force, tension and bending moment, q_t and q_n are the tangential and normal components of the externally applied load per unit length and K is the curvature of the elastic curve. Differentiation in (1-1) takes place along the arch length of the pipe as shown in Figure 2.

If $\psi(s)$ denotes the angle between the tangent at point S and the X-axis, then the curvature of the elastic curve is related to ψ through the equation.

$$\frac{d\psi}{ds} - k = 0 \qquad (1-2)$$

By virtue of the approximate theory - a generalization of the Euler - Bernouilli theory for beams we also have the connection between curvature and stress couple

$$M - EI\,k = 0 \qquad (1-3)$$

where E is Young's modulus of elasticity for the material and I the moment of inertia of the pipe cross section about the principal axis at its centroid. Introducing the horizontal and vertical components for stress resultants, H and V, referred to orthogonal cartesian coordinate system $_xA_y$, it follows,

$$H = T\cos\psi - N\sin\psi \qquad (1-4)$$
$$V = T\sin\psi + N\cos\psi$$

Likewise, for externally applied loads we obtain

$$q_H = q_t\cos\psi - q_n\sin\psi \qquad (1-5)$$
$$q_V = q_t\sin\psi + q_n\cos\psi$$

Substitution of equations (1-4) and (1-5) into equations (1-1) and by virtue of (1-2), leads into the set of equations,

$$\frac{dH}{ds} + q_H = 0 \qquad (1-6)$$

$$\frac{dV}{ds} + q_V = 0$$

$$\frac{dM}{ds} + V\cos\psi - H\sin\psi = 0$$

We proceed to integrate the first two equations of (1.6), in which case recalling (1.5),

$$H + \int_0^s q_h \, ds - H_1 = 0$$
$$V + \int_0^s q_v \, ds - V_1 = 0 \tag{1.7}$$

where H_1 and V_1 are the values of H and V at point A and where the tangent to the elastic curve coincides with the x-axis. Furthermore, since at the free end B the stress resultants H and V vanish, it follows that,

$$H_1 = \int_0^L q_h \, ds$$
$$V_1 = \int_0^L q_v \, ds \tag{1.8}$$

where L is the total length of the pipe.

Substitution of (1.8) into (1.7) yields

$$H = \int_0^L q_h \, ds$$
$$V = \int_0^L q_v \, ds \tag{1.9}$$

as functions implicitly of the arch length S, and integration takes place along the deformed centerline of the pipe. By virtue of (1.5), expressions (1.9) can also be written in the following form,

$$H = \int_0^L (q_t dx - q_n dy) \tag{1.10}$$
$$V = \int_0^L (q_t dy + q_n dx)$$

In deriving (1.10) the following identities were employed,

$$dx = \cos\psi ds \qquad (1.11)$$
$$dy = \sin\psi ds$$

and, the extension of the pipe has been neglected. If the angle of inclination is known as a function of arch length S then integration of (1.11) yields the coordinates of any point of the deflected pipe,

$$X(s) = \int_0^s \cos\psi(s)\, ds \qquad (1.11a)$$

$$Y(s) = \int_0^s \sin\psi(s)\, ds$$

The externally applied loads $q_t(s)$, $q_n(s)$, present in the system of equations (1.1), act along the tangential and normal direction to the deflected pipe respectively. The pipe deformation will be due to the presence of surface waves or ocean currents. Furthermore we assume that the deflection of the pipe takes place "quasistatically" in which case the inertia terms can be neglected. If the direction of motion of the water forms an angle α with the x-axis, as shown in (Figure 2) the normal component of the velocity to the deflected pipe will be equal to $\nu \sin(\alpha + \psi)$. Therefore, the hydrodynamic load per unit length of the deformed pipe will be,

$$q_1 = D\nu^2\sin^2(\alpha + \psi) \qquad (1.12)$$

where D is the drag coefficient. The coefficient D and velocity ν are taken constant along the pipe. The normal component of the load will then be

$$q_n = -q_o + D\nu^2\sin^2(\alpha + \psi) \qquad (1.13)$$

where q_o denotes the ground reaction and will be assummed here constant along the arch length S. The assumption is further made, that during deformation the direction of the soil reaction is not reversed. This is valid for the normal component and for most practical cases considered here. In the above discussion th angle α is always different from zero

$$\alpha > 0 \tag{1.14}$$

otherwise the pipe does not deform. We should also add the condition that when

$$\alpha = 0, \qquad q_n \equiv 0, \tag{1.15}$$

and without loss of generality it is assummed that the tangential component of the external load vanished, i.e.,

$$q_t = 0 \tag{1.16}$$

We like to stress at this point that deflection of the pipe will take place if and only if the distributed load q_1 is greate than the reacting frictional force q_o. This is tacitly assumed here, which case is also of greater practical importance. Evidently the magnitude of q_o depends on the nature of the ocean bottom, diameter of the pipe and other factors. However, it is included in our mathematical model.

Substitution of (1.13) and (1.16) into equations (1.5) yields the horizontal and vertical components

$$q_h = q_o \sin\psi - D\nu^2 \sin^2(\alpha+\psi) \sin\psi \tag{1.17}$$

$$q_v = -q_o \cos\psi + D\nu^2 \sin^2(\alpha+\psi) \cos\psi$$

The terms in (1.17) are modified in the following manner

$$\sin^2(\alpha+\psi)\sin\psi = \tfrac{1}{4}\{(2+\cos z\alpha)\sin\psi+\sin 2\psi\cos\psi - \sin(2\alpha+3\psi)\} \quad (1.18)$$

$$\sin^2(\alpha+\psi)\cos\psi = \tfrac{1}{4}\{(2-\cos z\alpha)\cos\psi+\sin z\psi-\cos(z\alpha+3\psi)\}$$

substitution of (1.17) into (1.7), carrying out the indicated operations we obtain

$$H(s)=H_1 - q_0 y + \frac{DV^2}{4}\{(2+\cos z\alpha)y+(\sin 2\alpha)x-\int_0^s \sin(2\alpha+3\psi)ds\}$$

$$V(s)=V_1 + q_0 y - \frac{DV^2}{4}\{(2+\cos 2\alpha)Y+(\sin z\alpha)y-\int_0^s \cos(2\alpha+3\psi)ds\} \quad (1.19)$$

In deriving (1.19), the differential relations (1.11) were employed. Inverting relations (1.4) we obtain

$$T(s) = H\cos\psi + V\sin\psi$$
$$N(s) = -H\sin\psi + V\cos\psi \quad (1.20)$$

where H(s), v(s), as in (1.19)

Likewise we may integrate the third equation of either (1.1) or (1.6) in order to obtain the bending moment at any point of the centerline of the pipe.

Integrating the third equation in (1.1) along the arch length if follows

$$M(s) = M_1 - \int_0^s N(s)ds = \int_s^L N(s)ds \quad (1.21)$$

Equation (1.21) has been derived by demanding that the resultant couple vanish at the free end B, i.e. at s=L and hence at support A

$$M_1 = \int_0^L N(s)ds \quad (1.22)$$

Substitution of (1.19) into (1.21), carrying out the indicated operations,

$$M(s) = M_1 + H_1 y - V_1 X - \frac{q_o}{2}(X^2+y^2)$$

$$-\frac{Dv^2}{4}\int\Big\{(2+\cos 2\alpha)\frac{y^2}{2}+(\sin 2\alpha)\,xy-\int_o^s \sin\psi\,ds\int_o^s \sin(2\alpha+3\psi)\,ds\Big\}$$

$$+\Big\{(2-\cos 2\alpha)\frac{x^2}{2}+(\sin 2\alpha)\,xy-\int_o^s \cos\psi\,ds\int_o^s \cos(\alpha+3\psi)\,ds\Big\} \qquad (1.23)$$

In deriving (1.23) the following relations were employed,

$$x\cos\psi ds = xdx = d\frac{(x^2)}{2}$$
$$y\sin\psi ds = ydy = d\frac{(y^2)}{2} \qquad (1.24)$$

$$y\cos\psi\ ds + x\sin\psi\ ds = y\ dx + xdy = d\ (xy)$$

By setting $M(L) = 0$ in (1.23) the couple M_1 can be evaluated and should be checked by (1.22).

The angle of inclination $\psi(s)$ being known, $x(s)$, $y(s)$ are evaluated through (1.11a)

By virtue of equations (1.2) and (1.3), equation (1.21) yields

$$K(s) = \frac{d\psi}{ds} = \frac{1}{EI}\int_s^L N(\psi(s))\,ds \qquad (1.25)$$

a first order equation for the angle $\psi(s)$ and where the integrand in (1.25) is given by

$$N(s) = V\cos\psi - H\sin\psi$$
$$= \cos\psi\int_s^L q_v\,ds - \sin\psi\int_s^L q_h\,ds \qquad (1.26)$$

In deriving equation (1.26) equations (1.4) were inverted and q_v (s), q_h (s) are given in (1.17). Evidently since the resultant couple vanishes at s=L so does the curvature of the elastic curve as is shown in (1.25). To the first order equation for ψ we must also adjoin the condition

$$\psi = 0 \qquad \text{at} \quad s=0 \qquad , \tag{1.27}$$

and integrating again follows

$$\psi(s) = \frac{1}{EI} \int_0^s d\sigma \int_\sigma^L N(\psi(\tau))\, d\tau \tag{1.28}$$

In expression (1.28) the flexural rigidity EI is assumed constant along s and N (ψ) as in (1.26). Equation (1.28) should be looked upon as the <u>non-linear integral equation</u> of the unknown dependent variable ψ(s). The solution of this equation if it exists will also constitute the solution to our problem in general.

In order to proceed further, we introduce the following nondimensional quantities,

$$\bar{q}_n = q_n / D\nu^2 = \bar{q}^\circ + \sin^2(\alpha+\psi), \tag{1.29}$$

$$\bar{q}_\circ = q_\circ / D\nu^2$$

$$[\bar{N}, \bar{T}] = [T, N] / D\nu^2 L$$

$$[\bar{H}, \bar{V}] = [H, V] / D\nu^2 L$$

$$\bar{M} = M/D\nu^2 L^2$$

$$\bar{K} = KL$$

$$[\bar{X}, \bar{Y}] = [X, Y] / L$$

$$U = \bar{S} = S/L \quad , \quad 0 \leq \bar{s} = U \leq 1$$

Hence expressions (1.17) take the form

$$\bar{q}_h = [\, \bar{q}_\circ \sin \psi - \sin^2 (\alpha + \psi) \sin \psi]$$

$$\bar{q}_v = [\, -q_\circ \cos \psi + \sin^2 (\alpha+\psi) \cos \psi \,] \quad (1.30)$$

Substitution of (1.30) into (1.9); (1.26), (1.21) and by (1.29)

$$\bar{H} = \int_u^1 \bar{q}_h du \quad , \quad (1.31)$$

$$\bar{V} = \int_u^1 \bar{q}_v du \quad ,$$

$$\bar{N} = \cos \psi \int_u^1 \bar{q}_v du - \sin \psi \int_\mu^1 \bar{q}_h du \quad ,$$

$$\bar{M} = \int_\mu^1 \bar{N} du \quad ,$$

$$\bar{X} = \int_\circ^\mu \cos \psi \, du \quad ,$$

$$\bar{Y} = \int_\circ^\mu \sin \psi \, du$$

Finally, equation (1.28) takes the form

$$\psi (u) = \mu \int_\circ^u d\sigma \int_\sigma^1 N (\psi(\tau)) d\tau \quad (1.32)$$

where the nondimensional parameter is given by

$$\mu = \frac{D \upsilon^2 L^2}{EI} \qquad (1.33)$$

It is equation (1.32) which governs the behavior of the pipe under the assummed load and support conditions. The solution to the equation (1.32) cannot be given in a closed form. However, although the integral equation for the dependent variable is a non-linear one, an iterative process may be employed, and hopefully convergent. Since the solution depends on the parameter μ in (1.33), the convergence or not of the iterative technique depends on the relative magnitude of μ. It should be expected that for small values of μ the iteration method will converge, for greater ones will not. Moreover in order to initiate the iterative scheme for a given value of μ, the proper choice of the initial shape of the deflected pipe ψ_0 (u), $0 \leq u \leq 1$, will influence the convergence behavior of the iterative scheme. It is the nature of non-linear problems, so that in seeking its solution, the results when obtained are restricted by the range of parameter-for instance μ - and the numerical technique used. Needless to say no numerical results can be derived but with the use of digital computers.

We like also to stress at this point that there exist for long bars pipes under normal to the deformed pipe loading, positions of unstable equilibrium. Since in general solutions, to non-linear problems are sought through linearization methods, such unstable positions of equilibrium will manifest themselves in numerical techniques used. Thus if the scheme converges at all, the influence may appear through the rate of convergence. This is also shown in the next section of present investigation, where the solution to a particular problem is given in closed form in terms of elliptic integrals and the instability appears through the vanishing a Jacobian, (2,34).

2. "A PARTICULAR SOLUTION"

In the absence of water currents, the pipe will be in a straight line configuration. Therefore, it is the presence of normal load $q_n > 0$ which induces bending of the pipe and hence displacements from the straight line position. Moreover, when the tangential stress resultant T (s) is compressive, in general any numerical scheme employed for the solution of our problem will be unstable. In what follows, we present a particular solution to the bending of a long bar where the intrinsic difficulties of a non-linear problem are apparent.

Consider an initially straight bar of length L, whose one end A is fixed as shown in Figure 3. The other end B is pulled toward a point O, on the plane of the ocean bottom, with a force P. No other forces are present, such as soil reactions or drag forces due to water currents. Let ψ_L be the angle between the x-axis and the tangent to the deflected pipe at point B, and α the angle between the cable load P and the normal to the bar at B setting.

$$\psi_L + \alpha = \beta \tag{2.1}$$

and overall equilibrium yields the following set of equations,

$$\Sigma V = N_A - P\cos\beta = 0$$

$$\Sigma H = T_A - P\sin\beta = 0$$

$$\Sigma M = M_A - P\,[\,X_L \cos\beta + Y_L \sin\beta\,] = 0 \quad (2.2)$$

In (2.2), N_A, T_A, M_A denote the reactive stress resultants at the fixed end A, and X_L, Y_L, the coordinates of end point B, about the set of cartesian coordinate axis X A Y as shown in Figure 3. Solution of (2.2) yields

$$N_A = P \cos\beta$$

$$T_A = P \sin\beta$$

$$M_A = P\,[X_L \cos\beta + Y_L \sin\beta\,] \quad , \quad (2.3)$$

where the tangential stress resultant component T_A is compressive. The bending moment at any point along the elastica will be given by

$$\begin{aligned} M(s) &= - M_A + N_A X + T_A Y \\ &= - P\,[(X_L - X) \cos\beta + (Y_L - Y) \sin\beta\,], \quad (2.4) \end{aligned}$$

where

$$\begin{aligned} X(s) &= \int_0^s \cos\psi \, ds \\ Y(s) &= \int_0^s \sin\psi \, ds \end{aligned} \quad , (2.5)$$

By virtue of the Bernouilli - Euler linear theory for the bending of beams it follows that

$$\begin{aligned} EI \frac{d\psi}{ds} &= M(s) \\ &= - P\,[X_L - X) \cos\beta + (Y_L - Y) \sin\beta] \quad , \quad (2.6) \end{aligned}$$

where E and I are the modulus of elasticity and moment of inertia of the cross section respectively. Differentiation of (2.6) with respect to the arch length s yields

$$EI \frac{d^2 \psi}{ds^2} = P \cos(\psi - \beta) \qquad , \qquad (2.7)$$

the differential equation governing the bending of the pipe shown in Figure 3. The solution of (2.7) must also satisfy the boundary conditions at A and B,

$$\psi = 0 \quad \text{at} \quad s = 0 \qquad , \qquad (2.8)$$

$$\frac{d\psi}{ds} = 0 \quad \text{at} \quad s = L \qquad ,$$

the second condition expressing the vanishing of the resultant couple at the free end. Introduce the transforma-- tions

$$\phi = \psi + \gamma = \psi - \beta + \frac{\pi}{2} \qquad , \qquad (2.9)$$

$$s = Lu \qquad ,$$

where u is a non-dimensional parameter and $0 \leq u \leq 1$. Substitution of (2.9) into (2.7) results in the equation

$$\frac{d^2\psi}{du^2} + \lambda \sin \phi = 0 \qquad , \qquad (2.10)$$

$$0 \leq u \leq 1$$

$$\gamma \leq \phi \leq \gamma + \psi_L = \phi_L \qquad ,$$

and the boundary conditions corresponding to (2.8)

$$\phi = \gamma \quad \text{at} \quad u = 0$$
$$\frac{d\phi}{du} = 0 \quad \text{at} \quad u = 1 \quad , \qquad (2.11)$$

and the parameter λ is given by

$$\lambda = \frac{PL^2}{EI} \quad , \qquad (2.12)$$

The order of the differential equation (2.10) can be reduced to one of the first order. This is accomplished by multiplying both sides of (2.10) by $\frac{d\phi}{du}$, integration and use of the second condition in (2.11), in which case we obtain

$$\frac{d\phi}{du} = \sqrt{\cos \phi - \cos \phi_L} \quad , \qquad (2.13)$$
$$0 \leq u \leq 1 \quad ,$$
$$\gamma \leq \phi \leq \phi_L \quad ,$$

Recalling that $u = \frac{s}{L}$ and also that during the deflection of the bar the total length of the elastica remains constant i.e. $s=L$, integration of (2.13) yields,

$$1 = \int_0^1 du = \frac{1}{\sqrt{2\lambda}} \int_0^{\phi_L = \gamma + \psi_L} \frac{d\phi}{\sqrt{\cos \phi - \cos \phi_L}}$$

$$= \frac{1}{2\sqrt{\lambda}} \left\{ \int_0^{\phi_L} \frac{d\phi}{\sqrt{\sin^2\frac{\phi_L}{2} - \sin^2\frac{\phi}{2}}} - \int_0^{\gamma} \frac{d\phi}{\sqrt{\sin^2\frac{\phi_L}{2} - \sin^2\phi}} \right\} \quad (2.14)$$

Let now

$$K = \sin \frac{\phi}{2} = \sin \left(\frac{\psi_L + \gamma}{2}\right) \quad , K \leq 1 \quad (2.15)$$

and select angle w such that

$$K \sin w = \sin \frac{\phi}{2} \quad , \quad (2.16)$$

and differentiation of (2.16) yields,

$$K \cos w \; dw = 1/2 \sqrt{1-k^2 \sin^2 w} \; d\phi \quad (2.17)$$

substitution of (2.15), (2.16) and (2.17) into (2.14) it follows

$$1 = \frac{1}{\sqrt{\lambda}} \left\{ \int_0^{\frac{\pi}{2}} \frac{dw}{\sqrt{1-k^2 \sin^2 w}} - \int_0^{w_1} \frac{dw}{\sqrt{1-k^2 \sin^2 w}} \right\} (2.18)$$

$$= \frac{1}{\sqrt{\lambda}} \{ K(k) - F(K, W_1) \}$$

where $K(k) = F(K \frac{\pi}{2})$ is the complete elliptic integral of the first kind and its value depends only on the modulus K. With the aid of (2.16), the variable w_1 has the value

$$w_1 = \sin^{-1} \left(\frac{\sin \frac{\gamma}{2}}{K}\right) \quad . \quad (2.19)$$

For a given value of the parameter λ in (2.12) and if W_1 is known, equation (2.18) determines the modulus K and hence the deflected shape of the bar. Proceeding in a similar manner as above it can be found that

$$\frac{s}{L} = \frac{1}{\sqrt{\lambda}} \left[F(K,W) - F(K,W_1) \right] \quad , \qquad (2.20)$$

where by (2.16) and (2.9)

$$w = \sin^{-1}\left(\frac{\sin\frac{\phi}{2}}{K}\right) = \sin^{-1}\left(\frac{\sin\left(\frac{\psi+\gamma}{2}\right)}{K}\right) \qquad (2.21)$$

We proceed now to evaluate the coordinates X(s), Y(s) of a point of the elastica as functions of the arch length s. Thus, by virtue of (2.5), (2.9) and (2.13), (2.16) it follows,

$$dx = ds\cos\psi = ds\cos(\phi-\gamma) = \frac{L}{\sqrt{\lambda}}\frac{dw}{(1-K^2\sin^2 w)^{1/2}}\cos(\phi-\gamma)$$

$$dy = ds\sin\psi = ds\sin(\phi-\gamma) = \frac{L}{\sqrt{\lambda}}\frac{dw}{(1-k^2\sin^2 w)^{1/2}}\sin(\phi-\gamma) \qquad (2.22)$$

But

$$\cos(\phi-\gamma) = (1-2K^2\sin^2 w)\cos\gamma + \sin\phi\sin\gamma$$

$$\sin(\phi-\gamma) = -(1-2K^2\sin^2 w)\sin\gamma + \sin\phi\cos\gamma \quad , \qquad (2.23)$$

substitution of (2.23) into (2.22) carrying out the indicated operations, we obtain the following results

$$\lambda\frac{X}{L} = [F(K,W_1) - F(K,W) + 2E(K,W) - 2\_(K,W_1)]\cos\gamma$$
$$+ 2K(\cos W_1 - \cos W)\sin\gamma$$

Electrical Engineering Department

$$\lambda \frac{Y}{L} = -[F(K,W_1)-F(K,W)+2E(K,W)-2E(K,W_1)] \sin\gamma + 2K(\cos W_1-\cos W)\cos\gamma \quad (2.24)$$

and W, W_1 are defined respectively in (2.16), (2.19). In (2.24) by $E(K,W)$ we denote the incomplete elliptic integral of the second kind. At the free end, the deflection is obtained, if in (2.24) we set $W=\frac{\pi}{2}$,

$$\frac{\lambda X_L}{L} = [F(K,W_1)-K(k)+2E(K)-2E(K,W_1)]\cos\gamma + 2K\cos W_1 \sin\gamma \quad , (2.25)$$

$$\frac{\lambda Y_L}{L} = [F(K,W_1)-K(k)+2E(k)-2E(K,W_1)]\sin\gamma + 2K\cos W_1\cos\gamma$$

From Figure 3 it is seen that

$$\tan\gamma = \frac{d_2-Y_L}{d_1-L+X_L} \quad , (2.26)$$

and equations (2.25) yield

$$\frac{\lambda}{L}[X_L\sin\gamma+Y_L\cos\gamma] = 2K\cos W_1 \quad , (2.27)$$

Substitution of (2.25) into (2.26) and by (2.27) relation (2.26) takes the form,

$$(\frac{d_1}{L}-1)\text{ sing }\gamma - \frac{d_2}{L}\cos\gamma + \frac{2K}{/\lambda}\cos W_1 = 0 \quad , (2.28)$$

which expresses the fact that the pulling load p at free end B is directed towards the point A. For the last term in (2.28),

$$\cos W_1, = \left(1 - \frac{\sin^2 \frac{\gamma}{2}}{K^2}\right)^{1/2} \qquad , \qquad (2.29)$$

where (2.19) has been used.

If the distance of free end B from point 0, is denoted by ρ_L then

$$\begin{aligned} \rho_L \cos\gamma &= d_1 - L + X_L \\ \rho_L \sin\gamma &= d_2 - Y_L \end{aligned} \qquad ; \qquad (2.30)$$

which yields the equation

$$\left(\frac{\rho_L}{L}\right)^2 - \left(-1 + \frac{d_1}{L} + \frac{X_L}{L}\right)^2 - \left(\frac{d_2}{L} - \frac{Y_L}{L}\right)^2 = 0 \qquad , \qquad (2.31)$$

The range for P_L should be taken between the limits

$$0 \leq \rho_L \leq \sqrt{d_1^2 + d_2^2} \qquad , \qquad (2.32)$$

The solution (2.20) and (2.24) contain the parameters λ, K, and γ as unknowns. For the complete solution of the particular case and for a given distance ρ_L these parameters are determined from the following requirements:

(i) that the total length of the bar will be equal to L and hence equation (2.18) follows,

(ii) that the load at B must go through point 0 and hence equation (2.28),

(iii) that ρ_L must be related to d_1, d_2, X_L, Y_L and L through equation (2.31).

Therefore, for $\frac{\rho_L}{L}$ known the trancedental equations (2.18), (2.28), (2.31) will yield the values for the parameters K, γ, λ, and hence the required solution for the deflected pipe. The solution of the system of three equations is intrinsically related to the Jacobian of the system, with respect to the same parameters. Therefore, the vanishing of the Jacobian of the system (2.33)

$$J\ (K,\ \gamma,\ \lambda) = 0 \qquad , \qquad (2.34)$$

is necessarily related to the stability of the present case under consideration.

Once the parameters K, γ, and λ have been determined for $J \neq 0$, the shape of the deformed bar is completely known as well as the stress resultants N (s), T(s) and stress couple M (s). Evidently the solution does not depend linearly to the externally applied load. The thus obtained solution is of great practical importance, for the case of existing pipe, whose free end is required to be moved to another position. Above solution will yield in that case the proper design data for given values of d_1, d_2, L and ρ_L.

SYMBOLS

X, Y, ...	orthogonal cartesian coordinates.
S, ...	arch length measured along the deflected pipe.
T, N, M, ...	Tangential stress resultant, sheer (normal) stress resultant, resultant bending moment.
H, V, ...	Stress resultants referred to the X, Y orthogonal coordinate system.
H^*, V^*, ...	Stress resultants referred to the ξ, n, orthogonal coordinate system.
L, ...	Length of deflected bar.
$\psi, \alpha, \beta, \gamma, \phi, \omega_1$, .	Angles measured in radians.
K, ...	Curvature of deflected centerline and in second part the modulus for elliptic functions.
$\bar{T}$, $\bar{N}$, $\bar{M}$, ...	Non-dimensional stress resultants.
q_t, q_n, ...	Tangential and normal to the deflected pipe applied loads.
q_H, q_V, ...	Applied loads when referred to X, Y axis
P, ...	Concentrated load applied to the free end of the pipe.
q_o, ...	Normal soil reaction component, constant.

D, ... Drag coefficient

v, ... Velocity of water current

E, ... Modulus of elasticity of pipe

I, ... Moment of inertia of cross section

$\mu = \frac{Dv^2L}{EI}$, ... Non-dimensional parameter,

$\lambda = \frac{PL^3}{EI}$, ... Non-dimensional parameter,

F (K,W), E(K). Complete elliptic integrals of the first and second kind.

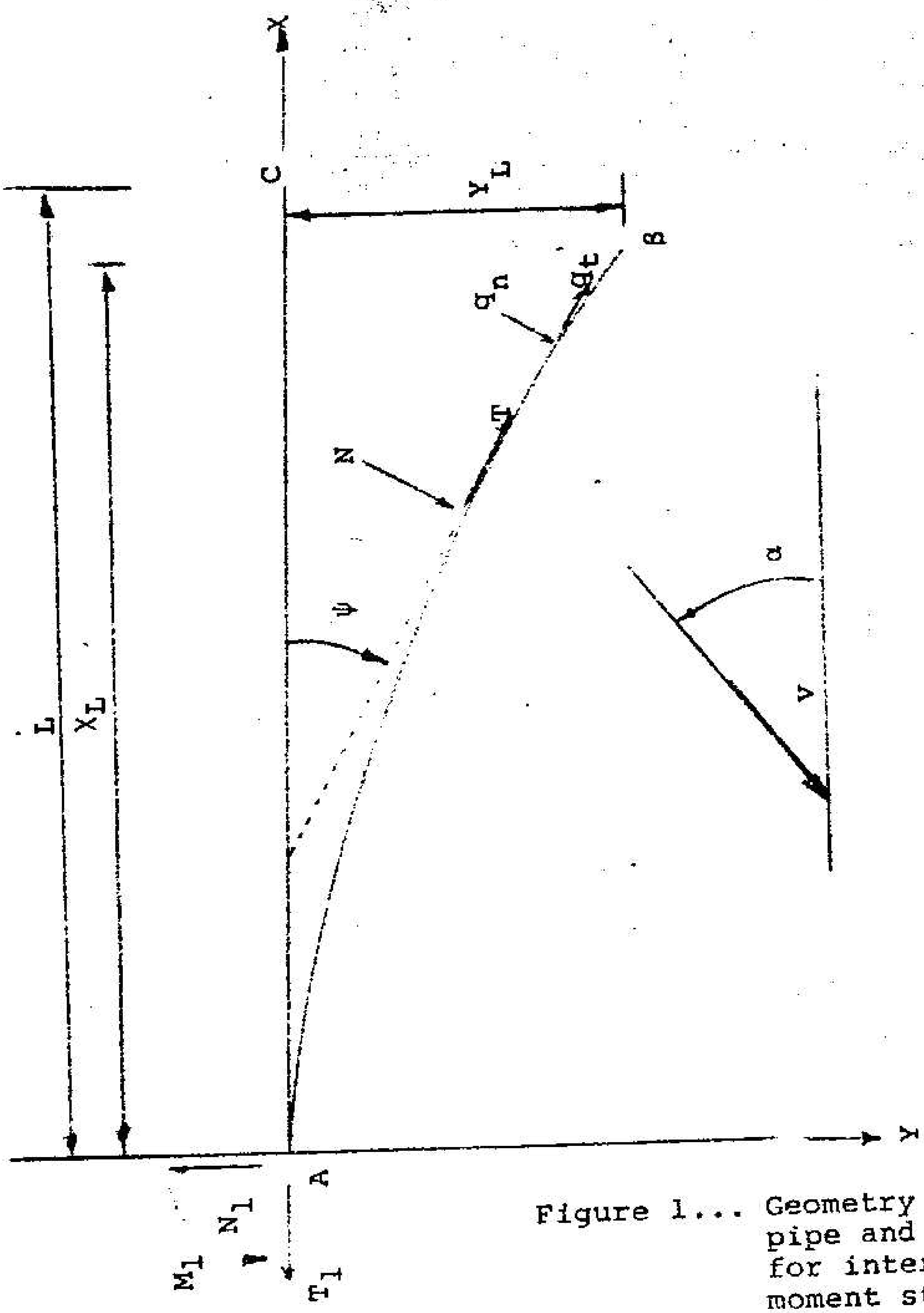

Figure 1... Geometry of the deflected pipe and positive directions for internal force and moment stress resultants.

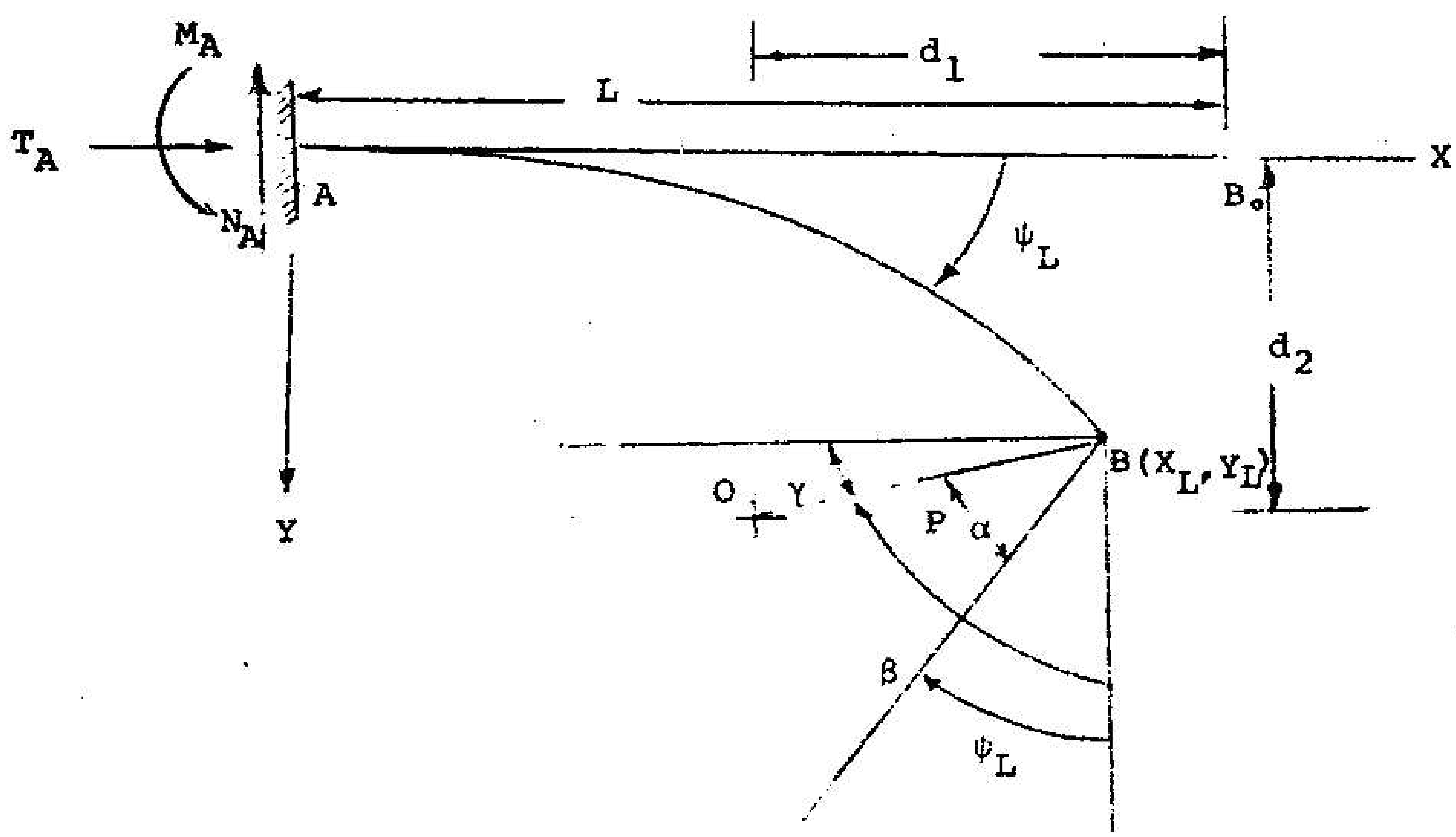

Figure 2...Geometry for deflected bar when pulled at free end by a concentrated force P passing through given point O.

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1.... Love A. E. H., "A Treatise on the Mathematical Theory of Elasticity", Dover Publications 1944.

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