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THEORETICAL AND EXPERIMENTAL STUDY OF THE SUPERCRITICAL FLOW IN CIRCULAR DRAINS AND THE FORMATION OF THE HYDRAULIC JUMP IN THE SAME

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VOLUME I

VOLUME I

TABLE OF CONTENTS

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CHAPTER 1

INTRODUCTION

1.1- DEFINITIONS AND CLASSIFICATION.- The Engineering works destined to conduct water from one place to another, are called water conduits. These, for their study, are classified into two groups:

1. Forced conduits
2. Open channels or free conduits

In the first ones, also called pipes, the water moves when submitted to a pressure superior to the atmospheric; in the second ones, the water circulates with a free surface, submitted to the atmospheric pressure, which can be considered as constant. So, in the first ones, the cross section of the conduit is completely full of water; in the second ones, the cross section of the conduit is partially with water. The limit between one and another type of conduit will be, when, being the cross section completely full, exists in the same atmospheric pressure. Fig. 1.1

The differential characteristic between the water movement by a forced conduit and by a free conduit is that, in the latters, the cross section of the stream flow is able to change according to the dynamic conditions present, while, in the forced conduits, the cross section is invariable, for being confined.

The free conduits are classified according to:

1. Their origin
2. Their kind of flow

The free conduits, by their origin, are classified into: natural and artificial, if they are product of nature, or if they are made by man.

The artificial free conduits can be of open cross section or of closed cross section. We have an example of the first one in an irrigation canal and of the second one, in a sanitary sewer, in a drain or in an acqueduct.

The free conduits of closed cross section are classified into: rectangular, square, circular, ovoid, horseshoe shape, elliptical, etc.

1.2 - CLASSIFICATION OF FREE CONDUITS ACCORDING TO THEIR FLOW.-

The free conduits according to their flow, can be classified, if we look at them from different points of view, that are here exposed:

A.- In relation to the characteristics of the stream flow through time. According to this criteria, the stream flow can have:

1.- Steady flow

2.- Unsteady flow

It is said that a stream flow has a steady flow, if the characteristics of the stream flow, through time, in a given point, do not change. Dealing with free stream flows, the characteristics are: the velocity of the stream flow and its depth. So that:

$$\frac{\partial V}{\partial t} = 0 \qquad \frac{\partial d}{\partial t} = 0$$

or in other words, if the velocity and the depth, in a given cross section, remain invariable through time.

The flow is unsteady, if the characteristics of the stream flow, velocity and depth, in a given cross section, change with time. So:

$$\frac{\partial V}{\partial t} \neq 0 \qquad \frac{\partial d}{\partial t} \neq 0$$

B.- In relation to the characteristics of the stream flow through space. According to this point of view, the stream flow can have:

1.- Uniform flow

2.- Varied flow

The stream flow has a uniform flow, if in a given instant, the velocity and the depth, along the stream flow, remain constant, or:

$$\frac{\partial V}{\partial s} = 0 \qquad \frac{\partial d}{\partial s} = 0$$

The flow of the stream is <u>varied</u>, if in a given instant, the velocity and the depth from one point to another, along the stream flow, vary, so:

$$\frac{\partial V}{\partial s} \neq 0 \qquad \frac{\partial d}{\partial s} \neq 0$$

According to the criteria (A) and (B), the stream flow in free conduits, can have:

1.- Steady and uniform flow
2.- Steady and varied flow
3.- Unsteady and uniform flow
4.- Unsteady and varied flow

The flow is steady and uniform, if the characteristics of the stream flow remain the same through time and space, so:

$$\frac{\partial V}{\partial t} = 0 \qquad \frac{\partial V}{\partial s} = 0 \qquad \frac{\partial d}{\partial t} = 0 \qquad \frac{\partial d}{\partial s} = 0$$

Therefore:

$$V = \text{constant} \qquad d = \text{constant}$$

We have an example of this flow, in practice, in an outfall of a sanitary sewerage that discharges under a constant head and that has enough length to establish a uniform flow.

The flow is steady and varied, if the characteristics of the stream flow do not vary with the time, although it does from one cross section to the other, along the stream flow, so:

$$\frac{\partial V}{\partial t} = 0 \qquad \frac{\partial V}{\partial s} \neq 0 \qquad \frac{\partial d}{\partial t} = 0 \qquad \frac{\partial d}{\partial s} \neq 0$$

We have an example of this flow, in practice, in the outlet of an outfall of a sanitary sewerage that discharges freely in a coast, in other words, that its outlet is not submerged in the sea.

The flow is unsteady and uniform, if the characteristics of the stream flow vary with the time, although remain the same in each cross section, in a given instant, along the stream flow, therefore:

$$\frac{\partial V}{\partial t} \neq 0 \qquad \frac{\partial V}{\partial s} = 0 \qquad \frac{\partial d}{\partial t} \neq 0 \qquad \frac{\partial d}{\partial s} = 0$$

The flow is unsteady and varied, if the velocity and depth of the stream flow vary as much through time as through space, so:

$$\frac{\partial V}{\partial t} \neq 0 \qquad \frac{\partial V}{\partial s} \neq 0 \qquad \frac{\partial d}{\partial t} \neq 0 \qquad \frac{\partial d}{\partial s} \neq 0$$

The most frequent flows in the practice of Hydraulic Engineering, are the steady uniform and the steady varied. Both flows are established in the circular conduits of the drains.

C. - In relation to the Reynolds number.- Another criteria for the classification of the flow of a free stream flow is the influence or effect of the inertial forces in relation to the viscosity forces.

The Reynolds number, defined by the expression:

$$N_R = \frac{V\ L}{\nu}$$

where:

N_R = Reynolds number

V = Mean velocity of the stream

L = Characteristic length of the phenomenon.

ν = Kinematic viscosity or the ratio between the absolute viscosity, μ, and the absolute density, ρ,

can be interpreted as the ratio between the iner ial forces and the viscosity forces.

In effect: the inertial and viscosity forces are given by the expressions:

$$F_v = \mu\ A \frac{dV}{dx} \qquad F_i = m \frac{dV}{dt} \quad \text{or} \quad F_i = \rho \forall \frac{dV}{dt}$$

where:

F_v = viscosity forces

A = considered area

V = velocity

x = distance perpendicular to the velocity

F_i = inertial force

m = mass

t = time

$\forall$ = volume

Both expressions presented in a dimensional way, we will have:

$$[F_v] = [\mu]\ L^2\ L\ T^{-1}\ L^{-1} \qquad [F_v] = [\mu]\ L\ [V]$$

$$[F_i] = [\rho]\ L^3\ L\ T^{-2} \qquad [F_i] = [\rho]\ L^2\ [V]^2$$

The ratio between the dimensions of both forces, is:

$$\frac{[F_i]}{[F_v]} = \frac{[\rho]}{[\mu]}\ \frac{L^2}{L}\ \frac{[V]^2}{[V]}$$

$$\frac{[F_i]}{[F_v]} = \frac{L\ [V]}{[\nu]}$$

Taking, as the characteristic lineal dimension, in this case, the hydraulic radius, R, or the relation between the wetted area and the wetted perimeter, and as the characteristic velocity, the mean velocity of the stream flow, we will have:

$$\frac{F_i}{F_v} = \frac{V\ R}{\nu} = N_R$$

The characteristic length, in the case of a forced conduit or pipe, is the diameter. In that case, the hydraulic radius is equal to $\frac{1}{4}$ of the diameter, or:

$$R = \frac{\frac{1}{4}\pi D^2}{\pi D} \qquad R = \frac{D}{4}$$

The expression of the Reynolds number dealing with a pipe, is:

$$N_R = \frac{V\ D}{\nu}$$

In the case of a pipe (forced conduits), the flows are classified according to the value of the Reynolds number, into:

1. - Laminar flow, if $N_R < 2,000$

2. - Transitional range, if $2,000 < N_R < 3,000$

3. - Turbulent flow, if $N_R > 3,000$

In order to relate the experiences with pipes with those experiences in open channels, we will have:

$$(N_R)_{CF} = \frac{4 R V}{\nu} \qquad (N_R)_{CL} = \frac{R V}{\nu}$$

Therefore:

$$(N_R)_{CF} = 4 (N_R)_{CL}$$

where the subindexes CF and CL mean forced conduits and open channels, respectively.

As we have seen in the above relation, we will have that the flow in open channels, will be:

1.- Laminar flow, if $N_R < 500$

2.- Transitional range, if $500 < N_R < 750$

3.- Turbulent flow, if $N_R > 750$

D.- In relation to the Froude number.- The flow of the streams in the free conduits, also is classified according to the relation between the inertial force and the gravity force.

The Froude number can be interpreted as the relation between the inertial and the gravity forces.

The Froude number has the expression:

$$N_F = \frac{V^2}{gL}$$

* Some hydraulicians define the Froude number by

$$N_F = \frac{V}{\sqrt{gL}}$$

where, g, is the intensity of the gravity or the force with which the earth attracts the mass unit.

The inertial and gravity forces are given, respectively, by the expressions:

$$F_i = m \frac{dV}{dt} \quad \text{or} \quad F_i = \rho \forall \frac{dV}{dt} \quad \text{and} \quad F_g = w \forall \quad \text{or} \quad F_g = \rho\, g \forall$$

Expressed in a dimensional way, we will have:

$$[F_i] = [\rho] L^3 \; L \; T^{-1} \; T^{-1} \qquad\qquad [F_g] = [\rho][g] L^3$$

Dividing the first formula by the second formula:

$$\frac{[F_i]}{[F_g]} = \frac{[\rho] \; L^2 \; [V]^2}{[\rho] \; [g] \; L^3}$$

$$\frac{[F_i]}{[F_g]} = \frac{[V]^2}{[g] \; L}$$

Taking as characteristic length the depth of the stream flow and as characteristic velocity the mean velocity of it, we will have:

$$\frac{F_i}{F_g} = \frac{V^2}{g \; d} = N_F$$

The flows are classified, according to the value of the Froude number, into:

1.- Subcritical flow, if $N_F < 1$

2.- Critical flow, if $N_F = 1$

3.- Supercritical flow, if $N_F > 1$

Rhebock, and later Bakhmetef, designated the expression V^2/gd, with the name kinetic factor, which is not anything else than twice the quotient of the velocity head divided by the depth of the flow. In effect:

$$\frac{V^2}{g\,d} = 2\,\frac{V^2}{2\,gd} = 2\left[\frac{V^2}{2g} : d\right]$$

In other words, it represents twice the relation between the kinetic energy and the potential energy of the flow.

When, in the mass of a fluid is produced a disturbance, or an alteration of the balance, this is transmitted to the remaining particles, intervening in the propagation, the gravity, elastic, superficial tension and viscosity forces, that act upon the disturbed particles and upon all particles around them.

In the case that these variations occur and spread in an ordered way, the propagation mechanism is named wave and according to the intervening (or predominant) force in the propagation, are named, in general, gravity, elastic, of superficial tension and viscosity waves.

In the Wave Mechanics, the wave celerity, its velocity of propagation, is given by the expression:

$$c = \sqrt{g\,d}$$

when it deals with small gravity waves, which occur in relatively shallow waters, as the waves presented in free conduits, when there occur momentary changes of the local depth of the water. Such momentary changes can be produced by the alterations or obstacles in the conduit, which causes the displacement of the water over and under the mean surface level, crea-

ting a wave that exerts a gravity force. It should be observed that the gravity waves can spread upstream with subcritical flow, but not in the supercritical flow, because $c > v$, in the first case; and $c < v$ in the latter case. So, this possibility or impossibility of spreading of the gravity wave can be used as a criteria in distinguishing the flow of the stream.

As in most of the free conduits the effects of gravity are the ones that control the phenomenon, the similarity law, for tests or experimental purposes, should be the constancy of the Froude number.

According to the (C) and (D) criteria, the flows in free conduits, can have:

1.- Subcritical laminar flow

2.- Subcritical turbulent flow

3.- Supercritical laminar flow

4.- Supercritical turbulent flow

1.3- OBSERVATIONS IN RELATION TO THE CLASSIFICATION OF THE FLOWS ACCORDING TO THE VARIATIONS OF THEIR CHARACTERISTICS IN TIME AND SPACE .- We have seen that, the characteristics of a flow through a free conduit, are the velocity of the flows and its depth. Both are time and space functions. So:

$$V = f_1 \ (s, t) \qquad d = f_2 \ (s, t)$$

then:

$$dV = \frac{\partial V}{\partial s} ds + \frac{\partial V}{\partial t} dt$$

$$dd = \frac{\partial d}{\partial s} ds + \frac{\partial d}{\partial t} dt$$

Having in mind the values of the partial derivations, exposed above, we will have:

In the steady flow:

$$\frac{\partial V}{\partial t} = 0 \qquad \frac{\partial d}{\partial t} = 0$$

In the uniform flow:

$$\frac{\partial V}{\partial s} = 0 \qquad \frac{\partial d}{\partial s} = 0$$

In the unsteady flow:

$$\frac{\partial V}{\partial t} \neq 0 \qquad \frac{\partial d}{\partial t} \neq 0$$

In the varied flow:

$$\frac{\partial V}{\partial s} \neq 0 \qquad \frac{\partial d}{\partial s} \neq 0$$

In the steady uniform flow:

$$\frac{\partial V}{\partial t} = 0 \qquad \frac{\partial d}{\partial t} = 0 \qquad \frac{\partial V}{\partial s} = 0 \qquad \frac{\partial d}{\partial s} = 0$$

If we substitute these values in the expressions for dV and dd, we will have:

$dV = 0$ and $dd = 0$

Therefore:

$V = constant$ and $d = constant$

In the steady varied flow:

$$\frac{\partial V}{\partial t} = 0 \qquad \frac{\partial d}{\partial t} = 0 \qquad \frac{\partial V}{\partial s} \neq 0 \qquad \frac{\partial d}{\partial s} \neq 0$$

If we substitute these values in the expressions for dV and dd, we will have:

$$dV = \frac{\partial V}{\partial s} ds \qquad dd = \frac{\partial d}{\partial s} ds$$

so:

$$V = f_1 (s) \qquad d = f_2 (s)$$

In the unsteady uniform flow:

$$\frac{\partial V}{\partial t} \neq 0 \qquad \frac{\partial V}{\partial s} = 0 \qquad \frac{\partial d}{\partial t} \neq 0 \qquad \frac{\partial d}{\partial s} = 0$$

If we substitute these values in the expressions for dV and dd, we will have:

$$dV = \frac{\partial V}{\partial t} dt \qquad dd = \frac{\partial d}{\partial t} dt$$

so:

$$V = f_1 (t) \qquad d = f_2 (t)$$

In the unsteady varied flow:

$$\frac{\partial V}{\partial t} \neq 0 \qquad \frac{\partial d}{\partial t} \neq 0 \qquad \frac{\partial V}{\partial s} \neq 0 \qquad \frac{\partial d}{\partial s} \neq 0$$

therefore:

$$V = f_1 (s, t) \qquad d = f_2 (s, t)$$

- Classification of free conduits
  - Origin
    - Natural
    - Artificial
      - Opened
      - Closed
        - Rectangular
        - Square
        - Horseshoe
        - Elliptical
        - Circular
          - Drains
          - Sanitary Sewerage
          - Acqueducts
  - Flow
    - V and d characteristics through time
      - Steady flow
      - Unsteady flow
    - V and d characteristics through space
      - Uniform flow
      - Varied flow
    - Reynolds number
      - Laminar flow
      - Transitional range
      - Turbulent flow
    - Froude number
      - Subcritical flow
      - Critical flow
      - Supercritical flow

CHAPTER 2

GEOMETRIC ELEMENTS OF THE CROSS SECTIONS OF OPEN CHANNELS

2.1 - GEOMETRIC ELEMENTS.- Geometric elements of the cross section of a free conduit or open channel are those properties of the section of the conduit, perpendicular to its bottom, which can be completely defined by the geometry of the section and by the depth of the flow.

These elements, that we will define, are the following:

1.- Depth of the flow, d

2.- Depth of the vertical section, y

3.- Wetted area, A

4.- Wetted perimeter, p

5.- Hydraulic radius, R

6.- Top width, B

7.- Mean depth, d_m

1.- The depth of the flow is the height of the perpendicular section of the direction of the flow.

2.- The depth of the vertical section is the height of the vertical cross section.

The relation between both depths is given by the formula:

$$d = y \cos \theta$$

In the case of small slopes both depths can be used indistinctively, because the cos θ tends to 1. Fig. 2.1.

3.- The wetted area is the area of the cross section of the flow, perpendicular to its direction.

4.- The wetted perimeter is the portion of the perimeter of the section of the flow, perpendicular to its direction, in contact with the conduit.

5.- The hydraulic radius is the quotient of the division of the wetted area by the wetted perimeter. The relation between the three elements will be:

$$R = \frac{A}{p}$$

6.- The top width is the width of the free surface of the flow.

7.- The mean depth is the quotient of the division of the wetted area by the top width, or:

$$d_m = \frac{A}{B}$$

2.2- WETTED AREA IN THE FLOW SECTION - Fig. 2.2 represents the cross section of a circular open channel, perpendicular to the direction of the flow, in which the water circulates to the depth of the flow, (d).

In order to calculate the wetted area, A, of the section, we should divide it into two parts: a circular sector, AOBC, and a triangle, AOB. So,

$$A = \text{sector area} - \text{triangle area}$$

but:

$$\text{Sector area} = \tfrac{1}{2}\ (\text{radius})\ (\text{arc's length})$$

where:

$$radius = \frac{D}{2}$$

$$arc's\ length = radius \times angle$$

Therefore:

$$A = \frac{1}{2} \times \frac{D}{2} \times \theta \times \frac{D}{2} - \frac{1}{2} \times 2 \times \overline{AD} \times \overline{OD}$$

but in the ACB triangle:

$$\overline{AD} = \frac{D}{2}\ sen \frac{\theta}{2} \qquad OD = \frac{D}{2}\ cos \frac{\theta}{2}$$

so:

$$A = \frac{D^2}{8}\ \theta - \frac{1}{2} \times 2 \times \frac{D}{2}\ sen \frac{\theta}{2} \times \frac{D}{2}\ cos \frac{\theta}{2}$$

Simplifying:

$$A = \frac{D^2}{8}\ \theta - \frac{D^2}{8}\ (2\ sen \frac{\theta}{2}\ cos \frac{\theta}{2})$$

but:

$$sen\ \theta = 2\ sen \frac{\theta}{2}\ cos \frac{\theta}{2} \qquad \text{(sine of twice the arc)}$$

therefore:

$$\boxed{A = \frac{D^2}{8} \left[\theta - sen\ \theta\right]}$$

As a proof of this, we should observe the following table. The previous formula does not have much practical use in Hydraulic Engineering, because it appeared as a function of the angle of the wetted section. It is more useful if the area appears as a function of the diameter of the conduit and of the depth of the wetted section.

TABLE.- Values of the wetted area for different values of the angle of the wetted section.

Angle Values		Value of sen θ	Value of the area
Degrees	Radians		
1	2	3	4
0	0	0	0
180	π	0	$1/2(\pi D^2/4)$
360	2π	0	$\pi D^2/4$

From Fig. 2.2 we obtain:

$$\overline{DC} = \overline{OC} - \overline{OD}$$

but:

$$DC = d \qquad y \qquad OC = \frac{D}{2}$$

also:

$$OD = AO \cos\frac{\theta}{2} = \frac{D}{2}\cos\frac{\theta}{2}$$

so:

$$d = \frac{D}{2} - \frac{D}{2}\cos \theta/2$$

from where:

$$\cos\frac{\theta}{2} = 1 - 2\frac{d}{D}$$

so:

$$\theta = 2\cos^{-1}\left[1 - 2\frac{d}{D}\right] \qquad \text{(Fórmula 2.1)}$$

but:

$$\text{sen}^2\frac{\theta}{2}\cos^2\frac{\theta}{2} = 1$$

from where:

$$\operatorname{sen}\frac{\theta}{2} = \sqrt{1 - \cos^2\frac{\theta}{2}}$$

or also:

$$\operatorname{sen}\frac{\theta}{2} = \sqrt{1 - \left[1 - 2\frac{d}{D}\right]^2}$$

$$\operatorname{sen}\frac{\theta}{2} = \sqrt{4\frac{d}{D} - 4\left[\frac{d}{D}\right]^2}$$

$$\operatorname{sen}\frac{\theta}{2} = \sqrt{4\frac{d}{D}\left[1 - \frac{d}{D}\right]} \quad \text{(Formula 2.2A)}$$

If we substitute the values of sen θ/2 and cos θ/2, in the formula sen θ, we will have:

tendremos:

$$\operatorname{sen}\theta = 2\sqrt{4\frac{d}{D}\left[1 - \frac{d}{D}\right]}\left[1 - 2\frac{d}{D}\right]$$

$$\operatorname{sen}\theta = 4\left[1 - 2\frac{d}{D}\right]\sqrt{\frac{d}{D}\left[1 - \frac{d}{D}\right]} \quad \text{(Formula 2.2)}$$

If we substitute the values of θ, given by Formula 2.1 and sen θ, given by Formula 2.2, we obtain:

$$A = \frac{D^2}{8}\left[2\cos^{-1}\left(1 - 2\frac{d}{D}\right) - 4\left(1 - 2\frac{d}{D}\right)\sqrt{\frac{d}{D}\left[1 - \frac{d}{D}\right]}\right]$$

or also:

$$A = \frac{D^2}{4}\cos^{-1}\left(1 - 2\frac{d}{D}\right) - \frac{D^2}{2}\left(1 - 2\frac{d}{D}\right)\sqrt{\frac{d}{D}\left(1 - \frac{d}{D}\right)}$$ Fórmula.(2.3)

This formula gives us the area of the wetted section as a function of the diameter of the conduit and of the depth of the flow. So that it will be of general use, we can present it in an adimensional way, dividing by the square of the diameter of the conduit. In this case, we can obtain:

$$\frac{A}{D^2} = \frac{1}{4}\cos^{-1}\left(1 - 2\frac{d}{D}\right) - \frac{1}{2}\left(1 - 2\frac{d}{D}\right)\sqrt{\frac{d}{D}\left(1 - \frac{d}{D}\right)}$$

If we make a table, calculating the values of A/D^2, for different values of d/D, it will be very useful. If we designate that value by K_A, we will have:

$$\frac{A}{D^2} = K_A$$

where, K_A is the value given by the list, which we will call Area factor. In any case, the area will be obtained from the formula:

$$A = K_A D^2$$

Analogically, we can make a chart in a double logarithmic paper, taking as the abcissas (independent variable) the values of d/D and as ordinates (dependent variable) the values of A/D^2. At the end of this chapter we have the table and the chart for the calculus of the wetted area, according to the previous formula.

2.3- WETTED PERIMETER IN THE FLOW SECTION- The wetted perimeter in a flow section, is given by the formula:

perimeter = arc = radius x angle

or:

$$p = \frac{D}{2}\,\theta$$

The value of the angle θ is given by formula 2.1, so:

$$p = D\cos^{-1}\left(1 - 2\frac{d}{D}\right) \quad \textbf{Formula (2-4)}$$

The adimensional correspondent formula is:

$$\frac{p}{D} = \cos^{-1}\left(1 - 2\frac{d}{D}\right)$$

Designating $\cos^{-1}\left[1 - 2\dfrac{d}{D}\right]$ by K_p, we will have:

$$\frac{p}{D} = K_p$$

and:

$$p = K_p\, D$$

If we make a table that gives the K_p value, for different values of d/D, it will be very useful, because we can calculate the value of d and of D. We will call the K_p factor, the wetted perimeter factor.

In the same way to what was done with the area factor, we could make a table or a diagram of its values as a function of the d/D relation.

2.4- HYDRAULIC RADIUS OF THE FLOW SECTION.- The hydraulic radius is given, in general, by the formula:

$$R = \frac{A}{p}$$

So, dealing with a flow section, partially full, the hydraulic radius will be obtained, dividing the value of the area, given by formula (2-3), by the value of the perimeter, given by formula (2-4), so:

$$R = \frac{\frac{D^2}{4} \cos^{-1}\left(1 - 2\frac{d}{D}\right) - \frac{D^2}{2}\left(1 - 2\frac{d}{D}\right)\sqrt{\frac{d}{D}\left(1-\frac{d}{D}\right)}}{D \cos^{-1}\left(1 - 2\frac{d}{D}\right)}$$

or:

$$R = \frac{K_A \ D^2}{K_p \ D}$$

Simplifying and designating K_A / K_p by K_R, we will have:

$$R = K_R \ D$$

or adimensionally:

$$\frac{R}{D} = K_R$$

The above derivation, tells us, that we can obtain <u>the factor of the hydraulic radius</u>, dividing the area factor, by the wetted perimeter factor, for the same values of d/D.

Now we are prepared to make a table or a diagram that gives us the value of the hydraulic radius factor, by the formula:

$$R = K_R\ D$$

2.5 - <u>TOP WIDTH IN THE FLOW SECTIONS</u>.- From figure 2.2, we have that:

$$B = 2\ \overline{AD}$$

$$B = 2 \times \overline{AO} \times sen\ \frac{\theta}{2}$$

$$B = D\ sen\ \frac{\theta}{2}$$

having in mind that AO = D/2.

If we substitute by sen θ/2, the value found before (Formula 2.2 A), we will have:

$$\boxed{B = D\sqrt{4\frac{d}{D}\left(1-\frac{d}{D}\right)}}$$

or in an adimensional way:

$$\boxed{\frac{B}{D} = \sqrt{4\frac{d}{D}\left(1-\frac{d}{D}\right)}}$$

The radical can be designated as top width factor which we will represent as K_B.

In a similar way, to what was done before, we can make a table or diagram that gives us the value of the top width factor, as a function of the d/D relation.

2.6 - MEAN DEPTH IN THE FLOW SECTIONS- We have seen that the mean depth is given by the expression:

$$d_m = \frac{A}{B}$$

or:

$$d_m = \frac{\frac{D^2}{4}\cos^{-1}\left(1 - 2\frac{d}{D}\right) - \frac{D^2}{2}\left(1 - 2\frac{d}{D}\right)\sqrt{\frac{d}{D}\left(1 - \frac{d}{D}\right)}}{D\sqrt{\frac{d}{D}\left(1 - \frac{d}{D}\right)}}$$

or also:

$$d_m = \frac{K_A\ D^2}{K_B\ D}$$

Making $K_A/K_B = K_m$, called mean depth factor, we will have:

$$d_m = K_m\ D$$

or in an adimensional way:

$$\frac{d_m}{D} = K_m$$

In a similar way to what was done before, we can make a table or diagram that gives us the value of K_m as a function of the d/D relation.

In the following pages we give some tables that are useful in calculating the geometric elements of the flow sections partially full. They have been taken from The Handbook of Hydraulics of H.W. King and E.F. Brater, edited by Mc Graw Hill Book Co., 5th. edition. The diagram has been taken from the work of E.A. Elevatorski, Hydraulic Energy Dissipators, edited by Mc Graw Hill Book Co..

TABLE No. 1 - Values of K_A, in the formula $A = K_A D^2$, for different values of the d/D relation.

d/D	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.0000	.0013	.0037	.0069	.0105	.0147	.0192	.0242	.0294	.0350
.1	.0409	.0470	.0534	.0600	.0668	.0739	.0811	.0885	.0961	.1039
.2	.1118	.1199	.1281	.1365	.1449	.1535	.1623	.1711	.1800	.1890
.3	.1982	.2074	.2167	.2260	.2355	.2450	.2546	.2642	.2739	.2836
.4	.2934	.3032	.3130	.3229	.3328	.3428	.3527	.3627	.3727	.3827
.5	.393	.403	.413	.423	.433	.443	.453	.462	.472	.482
.6	.492	.502	.512	.521	.531	.540	.550	.559	.569	.578
.7	.587	.596	.605	.614	.623	.632	.640	.649	.657	.666
.8	.674	.681	.689	.697	.704	.712	.719	.725	.732	.738
.9	.745	.750	.756	.761	.766	.771	.775	.779	.782	.784

<u>TABLE No. 2</u> - *Values of K_R, in the formula $R = K_R D$, for different values of the d/D relation.*

d/D	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.000	.007	.013	.020	.026	.033	.039	.045	.051	.057
.1	.063	.070	.075	.081	.087	.093	.099	.104	.110	.115
.2	.121	.126	.131	.136	.142	.147	.152	.157	.161	.166
.3	.171	.176	.180	.185	.189	.193	.198	.202	.206	.210
.4	.214	.218	.222	.226	.229	.233	.236	.240	.243	.247
.5	.250	.253	.256	.259	.262	.265	.268	.270	.273	.275
.6	.278	.280	.283	.284	.286	.288	.290	.292	.293	.295
.7	.296	.298	.299	.300	.301	.302	.302	.303	.304	.304
.8	.304	.304	.304	.304	.304	.303	.303	.302	.301	.299
.9	.298	.296	.294	.292	.289	.286	.283	.279	.274	.267

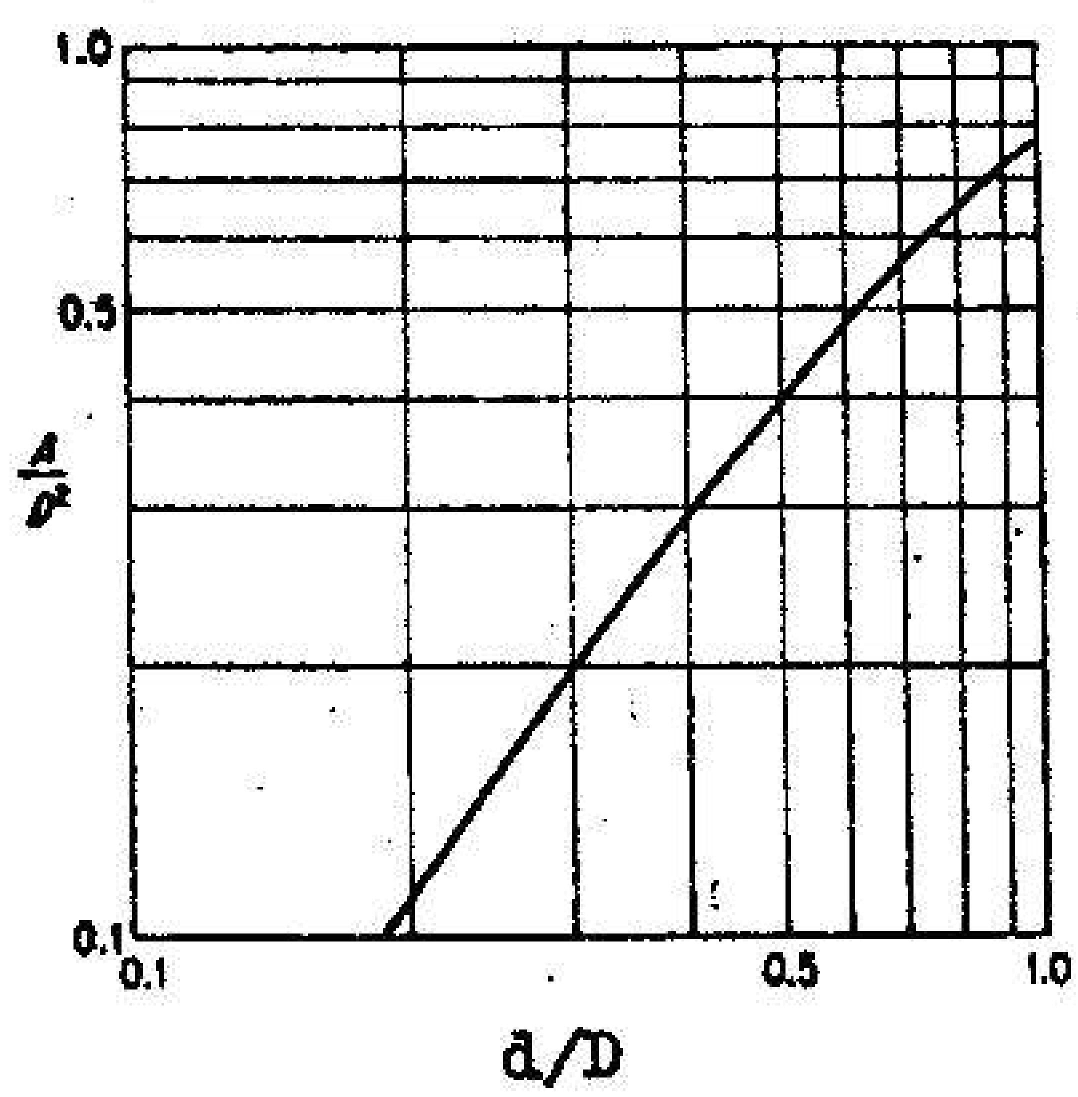
<u>TABLE No. 3</u> - *Values of K_B, in the formula $B = K_B D$, for different values of the d/D relation.*

d/D	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.000	.199	.280	.341	.392	.436	.475	.510	.543	.572
.1	.600	.626	.650	.673	.694	.714	.733	.751	.768	.785
.2	.800	.815	.828	.842	.854	.866	.877	.888	.898	.908
.3	.917	.925	.933	.940	.947	.954	.960	.966	.971	.976
.4	.980	.984	.987	.990	.993	.995	.997	.998	.999	1.000
.5	1.000	1.000	.999	.998	.997	.995	.993	.990	.987	.984
.6	.980	.975	.971	.966	.960	.954	.947	.940	.933	.925
.7	.917	.908	.898	.888	.877	.866	.854	.842	.828	.815
.8	.800	.785	.768	.751	.733	.714	.694	.673	.650	.626
.9	.600	.572	.543	.510	.475	.436	.392	.341	.280	.199

TABLE No. 4 - Values of K_m, in the formula $d_m = K_m\,D$, for different values of the d/D relation. (*)

d/D	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.000	.007	.013	.020	.027	.034	.040	.047	.054	.061
.1	.068	.075	.082	.089	.096	.103	.111	.118	.125	.132
.2	.140	.147	.155	.162	.170	.177	.185	.193	.200	.208
.3	.216	.224	.232	.240	.249	.257	.265	.274	.282	.291
.4	.299	.308	.317	.326	.335	.345	.354	.363	.373	.383
.5	.393	.403	.413	.423	.434	.445	.456	.467	.478	.490
.6	.502	.514	.527	.540	.553	.566	.580	.595	.610	.626
.7	.641	.657	.674	.692	.710	.730	.750	.771	.793	.817
.8	.842	.869	.897	.928	.960	.996	1.035	1.078	1.126	1.180
.9	1.241	1.311	1.393	1.492	1.613	1.768	1.977	2.282	2.792	3.940

DIAGRAM No. 1 - Values of K_A, in the formula $A = K_A\,D^2$, for different values of the d/D relation.



* $d_m = A/B$ is really the mean depth, only when the circular conduit works half full or with less depth.

CHAPTER 3

APPLICATION OF THE BERNOULLI THEOREM TO OPEN CHANNELS

3.1 - TOTAL HEAD IN FLOW CROSS SECTION. - The Bernoulli sum or total head corresponding to a point of the flow cross section, has the following expression:

$$H = z + \frac{p}{w} + \frac{V^2}{2g}$$

where:

- z, is the distance from the considered point, in the flow cross section, to a datum plane,
- p, the hydrostatic pressure existing in that point,
- V, the velocity of the water when it passes through that point,
- w, specific absolute weight of the water
- g, intensity of the gravity.

The Bernoulli sum or, total head for the flow cross section of a stream, through an open and straight channel, is:

$$H = z + d \cos \theta + \alpha \frac{V^2}{2g}$$

where:

- z, is the inverted distance of the conduit to the datum plane,
- d, is the depth of the flow,
- θ, is the angle that forms the gradient of the invert with the horizontal,
- V, the mean velocity in the cross section,
- α, a coefficient of correction.

In effect, suppose an open channel with a big slope invert, Fig. 3.1. Let us designate by L, the distance between the two sections that are considered, measured along the invert and x, the horizontal projection of the previous distance. From Fig. 3.1, we obtain:

$$z_1 - z_2 = L \text{ sen } \theta \qquad \text{sen } \theta = \frac{z_1 - z_2}{L}$$

This value receives the name of <u>hydraulic invert slope</u>.

$$z_1 - z_2 = x \tan \theta \qquad \tan \theta = \frac{z_1 - z_2}{x}$$

This value receives the name of <u>topographic invert slope</u>.

3.2 - <u>PRESSURE HEAD</u>.- Let us consider , Fig. 3.1, an elemental superficial element of area, dA, in the invert. If the pressure over this element is p, the total force over the surface, dA, will be:

$$dF_p = p\ dA$$

but the force, dF_p, is only the component of the weight of the oblique parallelepiped, of dA base and d height, perpendicular to the element, dA.

Volume of the oblique parallelepiped $d\forall = d.dA$

Weight of the previous parallelepiped $dW = w\ d\ dA$

Component of the weight perpendicular to dA,

$$dF_p = w\ d\ dA \cos\theta$$

Therefore:

$$p\ dA = w\ d\ dA \cos\theta$$

from where:

$$\frac{p}{w} = d \cos\theta$$

but:

$$d = y \cos\theta$$

Therefore:

$$\frac{p}{w} = y \cos^2\theta$$

It should be observed, that, d, is the depth of the flow, or, the height of the flow cross section, perpendicular to the invert, and, y, the height of the vertical cross section.

3.3 - VARIATION OF THE PRESSURE.- If in a fluid mass the pressure varies directly with the depth, it is said that the pressure follows the Hydrostatic Law. The pressure in a point (1), is given by:

p_1 = pressure in point (1)

w = specific absolute weight in the fluid

y = depth of the point in relation to the free surface of the fluid. Fig. 3.2

In effect; let us consider the inside of a fluid mass, Fig. 3.2, a superficial element dA. Let us consider, also, the prism that is vertical to the element dA. Let us suppose that this prism is solidified and let us apply the static equilibrium formula to it,

$$F_v = 0$$

where:

F_v, vertical forces.

The only vertical forces that act upon are:

1.- The weight of the prism: $dW = w\ y\ dA$

2.- Force due to the pressure: $dP = p\ dA$

Therefore:

$$p\ dA - w\ y\ dA = 0$$

from where:

$$p = w\ y$$

The Fig. 3.4, represents the Law of Variation of the Pressure with the Depth. As it is the law of variation of a no-flowing fluid, it is called Hydrostatic Law.

Let us consider, now, a flowing fluid mass, with a free surface. In this case, the distribution of the pressure follows the hydrostatic law when, only, does not exist any component of acceleration in the plane of the longitudinal section. This case, is presented, only, when fluid particles move as straight and parallel trajectories.

In effect, let us consider two cases:

1.- Motion as curvilinear trajectories

2.- Motion as converging or diverging trajectories.

Curvilinear convex trajectories.- Let us suppose a flowing fluid with curvilinear convex trajectories, Fig. 3.5, with a curvature radius, R. Let us consider the prism solidified above the superficial element dA. Because of the curvilinear trajectories, there will exist an acceleration and an inertial force (centripetal force), besides the weight of the prism and the reaction over its base.

Applying, in this case, the formula:

$$\Sigma F = m\, a$$

where:

ΣF = acting forces

m = mass

a = acceleration

So:

$$dP - dW = -\, dF_c$$

but:

$$dP = p\, dA \qquad dW = w\, y\, dA \qquad dF_c = \frac{w\, y\, dA}{g}\,\frac{V^2}{R}$$

Substituting:

$$p\, dA - w\, y\, dA = -\,\frac{w\, y\, dA}{g}\,\frac{V^2}{R}$$

Simplifying and clearing up p, we will have:

$$p = wy - \frac{w\, y}{g}\,\frac{V^2}{R}$$

Fig. 3.6, represents the Law of Variation of the Pressure in this case.

Curvilinear concave trajectories - Following the former procedure and having in mind the acting forces and their symbols, we will have:

$$dP - dW = dF_c$$

where:

$$dP = p\ dA$$

$$dW = w\ y\ dA$$

$$dF_c = \frac{w\ y\ dA}{g}\ \frac{V^2}{R}$$

therefore:

$$p\ dA - wy\ dA = \frac{w\ y\ dA}{g}\ \frac{V^2}{R}$$

Simplifying and clearing up p:

$$p = w\ y + w\ y\ \frac{V^2}{gR}$$

Fig. 3.8 represents the Law of the Variation of the Pressure in this case.

Diverging trajectories.- Let us consider, now, a diverging flow. Fig. 3.9. The acceleration produced by the variation of the velocity, can produce an inercial force that can have an effect on the variation of the pressure in the section.

The forces that act upon the elemental prism are:

Its weight: $dW = wy\ dA$

Reaction in the base: $dP = p\ dA$

Inertial force: $$dF_i = \frac{w\, y\, dA}{g}\, a_y$$

where a_y, is the vertical acceleration, in the section.

Therefore, applying the formula:

$$\Sigma F_y = m\, a_y$$

we will have:

$$p\, dA - w\, y\, dA = \frac{w\, y\, dA}{g}\, a_y$$

simplifying and clearing up a,p, we will have:

$$p = wy + wy\,\frac{a_y}{g}$$

<u>Converging trajectory</u>.- If we consider, now, a converging trajectory, Fig. 3.10, and we apply the same procedure, we will obtain:

$$p\, dA - w\, y\, dA = -\frac{w\, y\, dA}{g}\, a_y$$

Simplifying and clearing up p:

$$p = w\, y - \frac{a_y}{g}$$

So, in the case of converging and diverging flows, the effect of the inertial force is practically worthless, because there is little acceleration.

3.4 - <u>VELOCITY HEAD</u>. - The kinetic energy of every pound of water that passes through every point of the flow cross section of the stream, is given by:

$$\frac{V^2}{2g}$$

where V is the point velocity.

The kinetic energy of the amount of water that passes through the element of the area, dA, of the flow cross section, where the velocity V exists, is obtained in the following way, Fig. 3.11.

Volume of the water that passes in time unit: $dQ = v\ dA$

Weight in such volume: $dW = w\ v\ dA$

Kinetic energy of that weight: $dE_c = w\ V\ dA\ \frac{V^2}{2g}$

Total kinetic energy of the flow:

$$E_c = \frac{w}{2g} \quad V^3\ dA$$

The kinetic energy of each pound, or, the head due to the velocity, is:

$$\frac{E_c}{W} = \frac{\frac{w}{2g}\int V^3\ dA}{w\ A\ V_m}$$

$$\frac{E_c}{W} = \frac{V^3 \quad dA}{2g\ A\ V_m}$$

where, V_m, is the mean velocity in the flow cross section.

But the velocity head of the flow, expressed as a function of the mean velocity, can be formulated as:

$$\alpha \frac{V_m^2}{2g}$$

so:

$$\alpha \frac{V_m^2}{2g} = \frac{\int V^3 \, dA}{2g \, A \, V_m}$$

from where:

$$\alpha = \frac{\int V^3 \, dA}{A \, V_m^3}$$

The need for a <u>coefficient of correction</u>, is due to the fact that the distribution of the velocity, in the flow cross section is not uniform. If it were, V would be constant in all the section and we would have:

$$\alpha = \frac{V^3 \int dA}{A \, V_m^3}$$

$$\alpha = 1$$

because $V = V_m$

As we see all this, the total head of a flow is given by:

$$z + d \cos \theta + \alpha \frac{V^2}{2g}$$

The coefficient of correction, α, called coefficient of Coriolis, is the factor that, if it is applied to the head due to the mean velocity, to obtain the real energy per pound that passes through the flow cross section.

Let us demonstrate now, that the coefficient of Coriolis is more than 1. In effect, we know that the coefficient of Coriolis is given by the expression:

$$\alpha = \frac{\int V^3 \, dA}{V_m^3 \, A}$$

The point velocity, Fig. 3.12, can be considered to be made up of the sum of the mean velocity, plus a variable increment ΔV, that can be positive, negative or zero, so that:

$$V = V_m + \Delta V$$

According to the definition of mean velocity (the one that when multiplied by the area of the flow cross section of the stream has as result the discharge that passes through the mentioned section) we will have:

$$A \, V_m = \int V \, dA$$

Substituting the value of V, as a function of the mean velocity:

$$A \, V_m = (V_m + \Delta V) \, dA$$

$$= \int V_m \, dA + \int V \, dA$$

$$= V_m \, A + \int V \, dA$$

Therefore:

$$\int V \, dA = 0$$

On the other hand:

$$\int V^3 \, dA = \int (V_m + \Delta V)^3 \, dA$$

$$= \int (V_m^3 + 3 V_m^2 \Delta V + 3 V_m (\Delta V)^2 + (\Delta V)^3) \, dA$$

$$= \int V_m^3 \, dA + 3 V_m^3 \int V \, dA + 3 V_m \int (\Delta V)^2 \, dA + \int (\Delta V)^3 \, dA$$

But we have seen that:

$$V \, dA = 0$$

and that $\int (\Delta V)^2 \, dA$, is an essentially positive integral, that we can describe as:

$$\int (\Delta V)^2 \, dA = \eta \, V_m^2 \, dA$$

In relation to the integral:

$$\int (\Delta V)^3 \, dA$$

is very small, because ΔV can be positive, negative or zero, so that, logically, we can suppose that it tends to be zero.

As we have seen, we will have:

$$\int V^3 \, dA = V_m^3 A + 3 V_m^3 \eta A$$

so that, the value of α, will be:

$$\alpha = \frac{V_m^3 A + 3 V_m^3 \eta A}{V_m^3 A}$$

so that:

$$= 1 + 3 \eta$$

This shows that the coefficient of Coriolis is more than 1.

As we have seen, the application of the Bernoulli Theorem, between two flow cross sections of a stream, has the expression:

$$z_1 + d_1 \cos\theta + \alpha_1 \frac{V_1^2}{2g} = z_2 + d_2 \cos\theta + \alpha_2 \frac{V_2^2}{2g} + h_f$$

where h_f is the loss of head between the two considered sections.

3.5 - EXPERIMENTAL DETERMINATION OF THE COEFFICIENT OF CORIOLIS.- The procedure that is usually followed for the determination of the coefficient of Coriolis is due to Prof. M. P. O'Brien. It is a graphic procedure, that consists of tracing over the flow cross section of the conduit, the curves of equal velocity obtained from the measurements in different points of the cross section of the conduit and of the determination, by means of a planimeter, the area between every two consecutive curves of equal velocity. The next step is to trace a mass curve, taking as abscissas the accumulated areas and as ordinates the corresponding velocities. The value of integral $\int V^3 \, dA$, is obtained calculating or dividing, by means of a planimeter, the area under the curve comprehended between the same, the extreme ordinates and the axis of the abcissas. Substituting in the formula:

$$\alpha = \frac{\int V^3 \, dA}{V_m^3 \, A}$$

we obtain the value of the coefficient of Coriolis or correction factor of the head due to the mean velocity.

Let us apply the procedure to a circular cross section. Fig. 3.13, represents the flow cross section of a circular conduit. There appear the curves of equal velocity corresponding to the velocities V_1, V_2, V_3, V_4 and V_5, that were obtained in topography the contour lines.

On a system of axises of rectangular coordinates, the accumulated areas that comprehend the two consecutive curves of equal velocity, are taken as abscissas, beginning with the peripheral curve of the conduit. In the extremes of every partial area, are traced ordinates that represent, in scale, the cubes of the velocities that correspond to each curve of equal velocity that imitates them. The area that covers the curve traced in that way, the axis of the abscissas and the extreme ordinates, give us the value of the integral:

$$A_c = \int V^3 \; dA$$

The quotient of dividing the area A_c by the product $A\,V_m^3$, gives us the value of α.

The coefficient of correction of the head due to the mean velocity does not need to be considered in many open channel problems, but in others, it should be considered when the head due to the velocity has the same magnitude as the other terms that are placed in the Bernoulli equation.

An application of the factor of correction of the velocity head is in the calculation of the loss of head in an open channel. The steady uniform flow is very rare and usually there is a difference between the mean velocities that exist in both extreme cross sections, in the ones that the loss of head is desired. If the distribution of the velocity is similar in both extreme sections, it makes that α has the same value in such sections. If the distribution of the velocity in the extreme sections is changed by the presence of curves, contractions or other obstructions, the α coefficients in both sections will be different.

If in the Bernoulli equation, applied to open channels:

$$z_1 + d_1 \cos \theta + \alpha_1 \frac{V_1^2}{2g} = z_2 + d_2 \cos \theta + \alpha_2 \frac{V_2^2}{2g} + h_f$$

we clear out h_f, we will have:

$$h_f = (z_1 + d_1 \cos \theta) - (z_2 + d_2 \cos \theta) - \left[\alpha_2 \frac{V_2^2}{2g} - \alpha_1 \frac{V_1^2}{2g}\right]$$

This equation will show that α should not be considered if the difference:

$$(z_1 + d_1 \cos \theta) - (z_2 + d_2 \cos \theta)$$

is big in relation to the change between the heads due to the velocity.

CHAPTER 4

LOSS OF HEAD IN THE STEADY UNIFORM FLOW

4.1 - CHARACTERISTICS OF THE STEADY UNIFORM FLOW.- The water moves in a *steady flow* in an open channel, when the *depth* and the *velocity* of the flow remain constant, through time, in each one of the cross sections of the conduit, without being necessary that both elements of the flow remain constant through the whole conduit.

The steadiness of the flow has as result, that the *flow* that passes through any cross section, has a fixed and determined value, and because there is no accumulation or disappearance of the flow between two consecutive cross sections of the conduit, the flow remain constant, through time, along the whole conduit. So that, the *equation of continuity* is fulfilled, that is written in its elemental way:

$$Q = A_1 V_1 = A_2 V_2 = \dots\dots\dots\dots\dots = A_n V_n$$

where, Q, is the flow that moves through the conduit; A and

V the area and velocity, respectively, in the determined section by the subindex.

The former expression, is just an application of what is called the principle of the Conservation of the Matter.

Now, the stream flow is uniform, if the velocity of the stream remain constant along the conduit, or, it is fulfilled that:

$$V_1 = V_2 = \ldots\ldots\ldots\ldots\ldots = V_n$$

As a result of this we will have that:

$$A = \text{constant}$$

along the whole conduit.

As, the open channels that we are studying, are artificial conduits, (made by mankind) and always have prismatic shape, the area of the cross section is exclusive function of the depth of the flow, in a given case. So then, the depth of the flow, (depth of the normal section to the general direction of the movement) will be constant along the whole conduit, and we will have that:

$$d_1 = d_2 = \ldots\ldots\ldots\ldots\ldots\ldots = d_n$$

where d, represents the depth of the flow, and the subindex determines the cross section. Fig. 4.1.

As a result of this, the free top of the water (top grade line) and the invert grade line of the conduit, will be parallel lines. If we designate by s_s and by s_0, the respective slopes of each one of the grade lines, we will have that:

$$s_s = s_o$$

If L, represents the distance between both considered sections, measured along the invert of the conduit, we will have that:

$$s_o = \frac{z_1 - z_2}{L} = \text{sen}\ \theta$$

where z_1 and z_2, are the lowest elevations of the considered sections.

The application of the Bernoulli Theorem, to the cross sections (1) and (2), is :

$$z_1 + d_1 \cos\theta + \alpha_1 \frac{V_1^2}{2g} = z_2 + d_2 \cos\theta + \alpha_2 \frac{V_2^2}{2g} + h_f$$

where:

z = elevation of the point

d = depth of the flow

V = mean velocity

θ = angle formed between the invert of the conduit and the horizontal plane

α = coefficient of Coriolis

The subindex defines the considered section.

Having in mind that:

$$d_1 = d_2 \qquad \text{and that} \qquad V_1 = V_2$$

and that $\alpha_1 = \alpha_2$, for having both sections the same shape and area, the former expression is reduced to:

$$z_1 = z_2 + h_f$$

or:

$$z_1 - z_2 = h_f$$

Or, the difference in elevation is the same as the loss of head. This occurs only in the steady uniform flow. But:

$$z_1 - z_2 = s_o L$$

consequently:

$$s_o L = h_f$$

or also:

$$s_o = \frac{h_f}{L}$$

By definition, h_f/L is the loss of head by the length unit, or the slope of the energy grade line, that we will designate by s_e.

So, in the steady uniform flow, it is fulfilled that:

$$s_s = s_o = s_e$$

or, that the three grade lines are parallel.

4.2 - FORMULAS FOR THE STEADY UNIFORM FLOW.- The hydraulicians of all times have been preocupied in obtaining formulas that give the loss of head in the open channels and forced conduits. Both essentially, are the same.

From all the proposed formulas, the most used in expressing loss of head are the quadratic and the exponential.

The first ones come from the hypothesis that the resistant unitary force is a function only, of the square of the velocity. From there, its name.

The second ones, are the consequence of the hypothesis that the resistant unitary force is a function of certain power of the hydraulic radius (area divided by perimeter) and of another power of the velocity, other than two.

Expressed mathematically, the two former hypotheses are:

$$\tau = k_1 \ V^2 \quad \text{for quadratic formulas}$$

$$\tau = k_2 \ \frac{V^n}{R^m} \quad \text{for exponential formulas}$$

where:

τ = force resistant to movement, by each area unit.

V = mean velocity in the section.

R = hydraulic radius in the section

k_1 and k_2, proportionality constants, unknown

n and m, unknown exponents

Let us suppose that Fig. 4.2, represents an open channel with a steady uniform flow. Let us suppose that the flow cross section is circular, although what we expose here is general.

The forces that act upon the mass of water that comprehend sections (1) and (2), are:

1.- the weight, W, of such mass of water, given by:

$$W = w \ A \ L$$

where:

w= specific absolute weight of the water (weight of the volume unit)

A = area of the flow cross section

L = distance between sections (1) and (2).

2.- The forces due to the hydrostatic pressures in sections (1-1) and (2-2), which we will designate as P_1 and P_2. The force due to the pressure is given, in general, by:

$$P = w\ h_o\ A$$

where, h_o = head over the center of the gravity of the section. As sections (1-1) and (2-2) have the same shape and area, A and h_o have the same values, respectively, for sections (1-1) and (2-2), so that $P_1 = P_2$.

3.- The force resistant to the movement, which we will designate as F_R. This force has as value the area of the lateral surface of the conduit, in contact with the water, by the resistant force by area unit, or:

$$F_R = \tau\ p\ L$$

where: p = wetted perimeter (perimeter of the cross section of the conduit, in contact with the water)

Applying the formula of the dynamic balance:

$$F = m\ a$$

in the direction of the movement, and having in mind that the motion is uniform, we will obtain:

$$P_1 + W \text{ sen } \theta - P_2 - F_R = 0$$

Simplifying:

$$W \text{ sen } \theta = F_R$$

or also:

$$w\ A\ L \text{ sen } \theta = \ p\ L$$

but:

$$\frac{A}{p} = R \qquad \text{(hydraulic radius)}$$

$$sen\ \theta = \frac{h_f}{L} = s_e$$

therefore:

$$\tau = w\ R\ s_e$$

Up to this moment, the previous formula is a rational formula, it was found by mathematic reasoning.

Ist. Hypothesis.- According to this hypothesis:

$$\tau = k_1\ V^2$$

therefore:

$$w\ R\ s_e = k_1\ V^2$$

from where:

$$V = \sqrt{\frac{w}{k_1}}\ \sqrt{R\ s_e}$$

Designating w/k_f by C, we will have:

$$V = C\sqrt{R\ s_e}$$

This expression will be an empirical-rational formula, because it is based upon the matter that it was obtained, partly from mathematic reasoning, and partly using a hypothesis based upon empirical knowledges. In order to use this formula, it is necessary to determine the value of C by experimentation.

Representative of the previous formula, as we have called it, quadratic formula, is Chezy formula, obtained by the french hydraulic Chezy, in the way exposed above, for open channels.

Some experienced men, are specialized in the determination of the coefficient, C, usually called coefficient of Chezy.

Among the proposed expressions to determine the value of C, the best known are: that of Bazin, french hydraulic, and that of Ganguillet and Kutter, swiss hydraulics. Such formulas are:

$$C = \frac{157.6}{1 + \frac{m}{\sqrt{R}}} \qquad \text{Bazin formula}$$

$$C = \frac{41.65 + \frac{0.00281}{s} + \frac{1.811}{n}}{1 + \left[41.65 + \frac{0.00281}{s}\right]\frac{n}{\sqrt{R}}} \qquad \text{Ganguillet and Kutter Formula}$$

We see that, C, in Bazin formula, is a coefficient, (not a constant, as it was exposed by the formula) that depends upon the value of the hydraulic radius, and of m, a roughness coefficient, that depends, at the same time, upon the material in which it is constructed or is made up the open channel, and on the state of superficial polish.

According to Ganguillet and Kutter formula, the coefficient, C, depends upon the hydraulic radius, upon the slope and n, a factor that depends upon the kind and state of the superficial friction between the water and the conduit.

A study of the original report of Ganguillet and Kutter, demonstrate that the presence of s, in the formula is due to their effort in including the observations made on the Mississippi River.

Later investigators have demonstrated that the exactness of these observations is uncertain and it is probable that without them, it should not be necessary to include s in the formula. Nevertheless, the effect of s over C, according to the formula is very small, except for small values of s. For values comprehended between 0.01 and 0.001, the variations of C are very small, and for slopes larger than 0.001 the value of C, calculated for s = 0.001, can be used, without falling into larger errors than those inherent in the formula.

For that case, the formula is reduced to:

$$C = \frac{44.4 + \frac{1.811}{n}}{1 + 44.4 \frac{n}{\sqrt{R}}}$$

Ganguillet and Kutter formula has been widely used in the United States and in foreign countries, and they are greatly experienced in the selection of the roughness coefficient, n.

2nd. Hypothesis.- If we accept that:

$$\tau = k_2 \frac{V^n}{R^m}$$

as the best representative of the phenomenon, we will have, that:

$$w\, R\, s_e = k_2 \frac{V^n}{R^m}$$

from where:

$$V = \frac{w^{1/n}}{k_2} \; R^{(m+1)/n} \; s_e^{1/n}$$

Making:

$$\frac{w^{1/n}}{k_2} = K \qquad \frac{m+1}{n} = x \qquad \frac{1}{n} = y$$

we will have:

$$V = K\, R^x\, s_e^y$$

This formula, which is also <u>empirical-rational,</u> is the kind called <u>exponential,</u> because in order to put it into practice, it is necessary to know the values of "x", "y" and "k". According to it, the value of k, depends exclusively, upon the material that makes up the friction surface of the conduit with the water and upon the smooth state of such surface. There are many formulas of this kind, but one of the best known and employed, is the Manning formula, that has the expression:

$$V = \frac{1.486}{n} R^{2/3} s_e^{1/2}$$

Factor n, is a roughness coefficient, of the same numerical value as n of the formula of Ganguillet and Kutter.

The reasons why the Manning formula is so widely used, are:

1.- It has a more simple expression than the Bazin and the Ganguillet and Kutter formulas.

2.- The inequality of the values of factor n, with those of the formula of Ganguillet and Kutter and the great experimental knowledge of the roughness coefficient, for very diverse cases, because of the frequent use in the past of the Ganguillet and Kutter formula. In Europe, Manning formula is known as Stricker formula.

3.- Manning formula has the advantage of applying logarithms, being of easy adaptation for demonstrations and calculus.

Manning formula, given originally in the decimal metric system, has this expression in that system:

$$V = \frac{1}{n} R^{2/3} s_e^{1/2}$$

If the value of the velocity would be given in feet/sec., we will have that in m/sec., it will be V/3.28 and if the radius is expressed in feet, in meters, it will be R/3.28. Substituting those values in Manning formula:

$$\frac{V}{3.28} = \frac{1}{n}\left[\frac{R}{3.28}\right]^{2/3} s_e^{1/2}$$

Simplifying, we will have:

$$V = \frac{1.486}{n} R^{2/3} s_e^{1/2}$$

It is convenient to call the attention to the fact that the formulas of Chezy, Bazin, Ganguillet and Kutter, and Manning, refer, strictly, to the steady uniform flow.

The quadratic formula, really, is a particular case of the exponential formulas. In effect,

Quadratic formula $V = C\sqrt{R\, s_e}$ or $V = C\, R^{1/2}\, s_e^{1/2}$

Exponential formula $V = K\, R^x\, s_e^y$

We see that, the first formula is a particular case of the second, in which:

$$x = 1/2 \qquad y = 1/2$$

If the exponential formula would have, for the exponents, the correct values, the value of K, would be a function, only of the material that makes up the surface in contact with the water and of its polish or roughness state.

4.3 <u>COMPARATIVE STUDY BETWEEN THE FORMULAS OF WEISBACH-DARCY AND OF MANNING</u>.- The Weisbach-Darcy formula, very well known in Fluid Mechanics, for the study of the forced conduits, has as expression the following:

$$h_f = f \frac{L}{D} \frac{V^2}{2g}$$

where:

- h_f = loss of head between two flow cross sections,
- f = roughness coefficient of W + D,
- D = diameter of the conduit,
- V = mean velocity in the cross section,
- g = intensity of the gravity

If we remember that the hydraulic radius, R, is equal to $D/4$ and in the previous formula D is replaced by $4R$, we will have:

$$h_f = f \frac{L}{4R} \frac{V^2}{2g}$$

If in Manning formula:

$$V = \frac{1.486}{n} R^{2/3} s_e^{1/2}$$

we substitute s_e by its value, h_f/L, we will have:

$$V = \frac{1.486}{n} R^{2/3} (h_f/L)^{1/2}$$

if we clear out h_f, we have:

$$h_f = \frac{n^2}{1.486^2} \frac{L}{R^{4/3}} V^2$$

Equalizing the two values of h_f :

$$f \frac{L}{4\ R} \frac{V^2}{2g} = \frac{n^2}{1.486^2} \frac{L}{R^{4/3}} V^2$$

and clearing out n, we will have:

$$n = 0.0926\ R^{1/6}\ f^{1/2}$$

The formula that relates the value of n, in Manning formula, with the value of f, in the Weisbach-Darcy formula.

The Weisbach-Darcy formula, in the present, has been widely studied, so much, from the theoretical to the experimental point of view, up to the point that we have drawn into the general conclusion of:

$$f = \Phi\ (N_R\ ,\ \frac{e}{D})$$

where:

Φ = function of,

N_R = Reynolds number

e/D = relative roughness

e = characteristic height of the roughness.

Dealing with forced conduits:

$$N_R = \frac{V\ D}{\nu} \qquad \text{relative roughness} = \frac{e}{D}$$

Dealing with open channels:

$$N_r = \frac{4\ V\ R}{\nu} \qquad \text{relative roughness} = \frac{e}{4R}$$

Therefore, dealing with open channels:

$$f = \varphi\left(\frac{4\ V\ R}{\nu} , \frac{e}{4R} \right)$$

therefore, we will have:

$$n = 0.0926\ R^{1/6} \varphi \left[\frac{4\ V\ R}{\nu} , \frac{e}{4R} \right]$$

that can be expressed, in a simpler way, saying that, n, in Manning formula should be a function of R, V, e and ν *or:*

$$n = \varphi\,(R, V, e, \nu)$$

As:

$$v = \frac{1.486}{n}\ R^{2/3}\ \ s_e^{1/2}$$

indicates us that dealing with a conduit that conducts water, with a determinated temperature, the exponents of V and R, in Manning formula, do not have, in general, the correct values. If those exponents would be correct, n, would vary, only with the kinematic viscosity, and with the relative roughness, $e/4R$.

We should remember, that from a study of Moody's diagram for the variations of the coefficient, f, as a function of the Reynolds number, N_R, and of the relative roughness, e/D, inside the turbulent flow, the following conclusions are drawn out:

1.- In those pipes that are hydraulically smooth, the value of f, depends, exclusively, upon the Reynolds number.

2.- In those pipes that are hydraulically rough, the value of f, depends, exclusively, upon the relative roughness.

3.- In the zone that comprehends the smooth pipes, and the rough pipes, the value of f, depends upon the value of the Reynolds number and the relative roughness.

If we examine the previous expression:

$$n = 0.0926\ R^{1/6}\,\varphi\left[\frac{4\ V\ R}{\nu}\ ,\ \frac{e}{4R}\right]$$

is logical to think that, in open channels will occur:

1.- In open channels hydraulically smooth, the value of n, will vary with the hydraulic radius and with the velocity.

2.- In the open channels, hydraulically rough, the value of n, will vary, also, with the hydraulic radius and with the velocity.

3.- In open channels that comprehend the smooth and the rough, the value of n, will vary with the hydraulic radius and the velocity.

We should observe that if we say that the value of n, will vary with the hydraulic radius, is the same thing as if we say that the exponent 2/3 in Manning formula, is not correct. Analogically, if we say that it varies with the velocity, is the same thing as if we say that the exponent of 1/2 is not correct either.

As the hydraulic radius vary with the depth of the flow, there is not any doubt that, according to the previous reasoning, the value of n, will vary with the depth of the flow.

CHAPTER 5

LOSS OF HEAD IN THE STEADY VARIED FLOW

5.1 - <u>ESTABLISHMENT OF THE UNIFORM FLOW</u>.- The steady uniform flow does not occur frequently in practice. Strictly talking, it never occurs in natural open channels. In artificial open channels, it occurs only when, having the conduit, a steady flow, it has also an inverted gradient in the direction of the movement and enough conduit length so that the mean velocity is constant along it. This is obtained when the component of the gravity action, in the direction of the movement, equals the force resistant to the movement if it is constant, because the physical factors of the conduit remain invariable, such as: shape of the cross sections, material that makes up the walls and bottom of the conduit, degree of roughness of that material and straight alignment of the conduit.

In the conduits with a null gradient (horizontal inverts) and in the adverse gradient, is impossible the existence of the steady uniform flow.

There exists the possibility, not certainty, that the steady uniform flow could be present, in the mild or subcritical gradients, in the critical and in the steep or supercritical, if there exists enough conduit length, to give occasion to the uniformity of the mean velocity along the conduit.

If the water enters in a conduit slowly, the velocity, and so, the resistance, is small, exceeding the gravity force to the resistance, what gives occasion to an accelerated circulation in the span of the entrance of the conduit. The velocity and the resistance increase gradually, until the equilibrium between the resisting force and the component of the gravity force is established. The portion or span necessary for the establishment of the uniform flow, is called transitional zone. If the conduit is shorter than the length required for this zone, the uniform flow can't be obtained.

With illustrative purposes, only, in Fig. 5.1, let us represent three cases of long open channels.

In Fig. 5.1 A, it is dealt with an open channel with a mild or subcritical inverted gradient. The top of the water in the entrance transitional zone, appears wavy. The flow is uniform, only, in the central span, and varied, in the entrance and exit spans. Theoretically, the variable depth in each extreme zone is near the normal depth gradual and asymptotically. For practical purposes, although, the depth can be considered constant if the variation of the depth, d, is within certain margin, let us say 1% of the normal depth, d_0.

Fig. 5.1 B, represents an open channel with a critical gradient. The top of the water in the critical uniform flow is unsteady. There can appear ondulations in this flow, although the average of the depths is constant and the circulation can be considered with a uniform flow.

In Fig. 5.1 C, is presented an open channel with a steep or supercritical gradient. There, the water passes, in the transitional zone, with a subcritical depth, gradually. When the transitional zone is passed, the flow approaches to the uniform. The length of the transitional zone depends upon the discharge and the physical conditions of the same, such as: shape of the entrance, gradient and roughness.

5.2 - CHARACTERISTIC OF THE STEADY VARIED FLOW. - The steady varied flow is presented in practice, frequently, in open channels.

We have seen, previously, that in the steady flow, as a consequence of the characteristic of that flow, constancy of the depth and the velocity through time, has as result, that the continuity equation should be fulfilled:

$$Q = A_1V_1 = A_2V_2 = \ldots\ldots\ldots\ldots\ldots = A_nV_n$$

The flow is varied, (in relation to the space), if the velocity of the flow varies along the conduit, or:

$$V_1 \neq V_2 \neq \ldots\ldots\ldots\ldots\ldots\ldots \neq V_n$$

Therefore; the flow is steady and varied, when both conditions exposed above occur, or:

$$A\ V = constant \qquad V = variable$$

As a consequence, we will have:

$$A = variable$$

or, that the wetted area is variable along the conduit.

Now, having in mind that the open channels that we are studying, are prismatic conduits and that the wetted area is a function of the depth of the water, only, this will vary from one section to the other, and we will have that:

$$d_1 \neq d_2 \neq \ldots\ldots\ldots\ldots\ldots\ldots \neq d_n$$

As a consequence of all what has been exposed, the top of the water (water surface gradient) and the invert gradient or bottom of the open channel, will not be parallel lines. Strictly, the gradient of the bottom of the open channel, is a straight line; the water surface gradient is a curve, that between two neighboring sections, can be considered as a straight one, because the tangents are confused with the curve between the two considered sections. Therefore:

$$s_o \neq s_s$$

where:

$$s_o = \frac{z_1 - z_2}{L} \quad \text{and} \quad s_s = \frac{z_1 + d_1 \cos \theta - z_2 - \cos \theta}{L}$$

Applying Bernoulli's Theorem sections (1) and (2), from

Fig. 5.2, we will have:

$$z_1 + d_1 \cos \theta + \alpha_1 \frac{V_1^2}{2g} = z_2 + d_2 \cos \theta + \alpha_2 \frac{V_2^2}{2g} + h_f$$

but:

$$d_1 \cos \theta + \alpha_1 \frac{V_1^2}{2g} = e_1 \quad \text{and} \quad d_2 \cos \theta + \alpha_2 \frac{V_2^2}{2g} = e_2$$

we will have:

$$\frac{z_1 - z_2}{L} + \frac{e_1 - e_2}{L} = \frac{h_f}{L}$$

or also:

$$s_o + \frac{e_1 - e_2}{L} = s_e$$

As:

$$\frac{e_1 - e_2}{L} \neq \theta \qquad s_o \neq s_e$$

Where:

$$s_e = \frac{\left[z_1 + d_1 \cos \theta + \alpha_1 \frac{V_1^2}{2g} \right] - \left[z_2 + d_2 \cos \theta + \alpha_2 \frac{V_1^2}{2g} \right]}{L}$$

Therefore, the three gradients are not parallel between themselves, because they have their respective unequal gradients, or:

$$s_o \neq s_s \neq s_e$$

In spite of what was exposed, we can say that:

In the steady and varied flow, the slope of the three gradients, that of the bottom of the open channel, the superficial and the one of energy, <u>are unequal between themselves, and those three gradients are not parallel lines each other with each one of the other two.</u>

5.3 - <u>LOSS OF HEAD IN THE STEADY UNIFROM FLOW</u>.- To determine the slope of the gradient of energy there is not any formula that gives its value directly. At present time, to obtain it, different hypotheses are made, that can be summarized in the following synoptic table:

Methods to determine the value of s_e

1. $$s_e = \frac{(s_e)_1 + (s_e)_2}{2}$$ where
   $$(s_e)_1 = \frac{V_1^2\, n^2}{1.486^2\, R_1^{4/3}}$$
   $$(s_e)_2 = \frac{V_2^2\, n^2}{1.486^2\, R_2^{4/3}}$$

2. $$s_e = \frac{V_m^2\, n^2}{1.486^2\, R_m}$$ where
   a)- $$A_m = \frac{A_1 + A_2}{2}$$
   $$p_m = \frac{p_1 + p_2}{2}$$
   b)- $$V_m = \frac{V_1 + V_2}{2}$$
   $$R_m = \frac{R_1 + R_2}{2}$$

In procedure (1) and in (2b), we have that:

$V_1 = Q/A_1$ $V_2 = Q/A_2$ $R_1 = A_1/p_1$ $R_2 = A_2/p_2$

In procedure (2a), we have that:

$$V_m = \frac{Q}{A_m} = \frac{2Q}{A_1 + A_2} = \frac{2Q}{\dfrac{Q}{V_1} + \dfrac{Q}{V_2}} = \frac{2\, V_1\, V_2}{V_1 + V_2}$$

$$R_m = \frac{A_1 + A_2}{p_1 + p_2}$$

The hypothesis in which procedure (1) is based, is the following: In the steady flow, gradually varied, the slope of the gradient of energy has as value the average of the values of the slopes of such gradient, if the same discharge would circulate to every depth that exist in the selected extr sections, with a steady uniform flow.

The hypothesis in which procedure (2) is based, is the following: In the steady flow, gradually varied, the slope of the gradient of energy has as value the one that correspond to the averages of the velocities and hydraulic radii in each one of the selected extreme sections.

Within this last hypothesis, can be distinguished two sub-cases. These depend upon the way in which the mean velocity and the mean hydraulic radius are obtained.

In subcase (a), it is supposed that:

$$R_m = \frac{A_m}{p_m} = \frac{\dfrac{A_1 + A_2}{2}}{\dfrac{p_1 + p_2}{2}} = \frac{A_1 + A_2}{p_1 + p_2}$$

In subcase (b), it is supposed that:

$$R_m = \frac{R_1 + R_2}{2} = \frac{\dfrac{A_1}{p_1} + \dfrac{A_2}{p_2}}{2} = \frac{A_1 p_2 + A_2 p_1}{2 p_1 p_2}$$

In subcase (a), it is supposed that:

$$V_m = \frac{Q}{A_m} = \frac{2\,Q}{A_1 + A_2}$$

In subcase (b), it is supposed that:

$$V_m = \frac{V_1 + V_2}{2} = \frac{\dfrac{Q}{A_1} + \dfrac{Q}{A_2}}{2} = \frac{Q}{2} \quad \frac{A_1 + A_2}{A_1 A_2}$$

CHAPTER 6

SUBCRITICAL, CRITICAL AND SUPERCRITICAL FLOWS

6.1 - SPECIFIC ENERGY OF CIRCULATION. - The specific energy of circulation of a free conduit, is the total energy of each pound of water that passes through a cross section of the flow, in relation to an horizontal plane that passes through the lowest point of the cross section.

In Fig. 6.1, is represented the longitudinal section of the circular drain. In one of the ends is represented the cross section.

According to the definition previously exposed, the specific energy is given by the following formula:

$$e = d \cos \theta + \alpha \frac{V^2}{2g}$$

In effect, let us consider, first, the particles of water that are found in AB axis.

The potential energy of a pound of water that is placed at any height over that axis, is equal to:

$$d \cos \theta$$

If the pound is located at the top of the water, its potential energy will be the vertical distance between C and the horizontal plane that passes through A, or:

$$AD = d \cos \theta$$

If the pound is located in A, the pressure head in that point, equivalent to a potential energy, is equal to:

$$AD = d \cos \theta$$

If the pound is located in E, will be submitted to a pressure head DF, plus a gravity head FA. The sum:

$$DF + FA = AD = d \cos \theta$$

Let us consider now, the particles of the water that are found in A'B' axis.

If the pound of water is found in C', it will be submitted to a gravity head equal to:

$$D'A' + A'G = D'G = AD = d \cos \theta$$

A similar analysis, for all the possible positions in the flow cross section, will give us the same result.

Therefore, the potential energy of a pound of water, for any position in the cross section will be given by:

$$d \cos \theta$$

The kinetic energy of a pound that passes through the flow cross section, as a function of its mean velocity and of the coefficient of distribution of the velocity or the Coriolis' coefficient, is given by:

$$\alpha \frac{V^2}{2g}$$

So, the specific energy, will be given by:

$$e = d \cos \theta + \alpha \frac{V^2}{2g} \quad \text{(Formula 6-1)}$$

If we have in mind that: V = Q/A, the previous formula is converted into:

$$e = d \cos \theta + \alpha \frac{Q^2}{2gA^2} \quad \text{(Formula 6.2)}$$

6.2 - ALTERNATING LEVELS OF EQUAL SPECIFIC ENERGY. - If we represent the previous equation in a graphic, in a system of coordinate axis, that has the specific energies as abscisas and the depths as ordinates, we will obtain a similar graph as Fig. 6.2. In the figure, three curves e-d are represented corresponding to three different discharges.

An anlysis of the previous graph representation, lead us to the following conclusions:

1. - Any discharge can circulate at two different depths, with the same content of specific energy. For example, the discharge Q_1, can circulate at d_1 and d_2 depths, with the same content of energy as e_1.

2. - For a given discharge, when the specific energy decreases, the values of both depths become nearer.

3. - When the content of specific energy, for a

given discharge, is minimum, both depths are joined into only one depth.

4 - The values of both depths, for a given discharge, that have the same content of specific energy, receive the name of alternative levels or depths of equal specific energy.

5 - The level or depth that corresponds to the minimum content of specific energy, is designated as the critical depth.

6 - Line OA has an angular coefficient m, equal to cos θ.

7 - The values of the critical depths increase as the discharges increase.

8 - The values of the critical depths, for the different discharges, fall over a straight line, or the following relation is fullfilled:

$$\frac{d_{c-1}}{d_{c-2}} = \frac{e_{min-1}}{e_{min-2}}$$

9 - The line OB, divides the different curves of specific energy in three regions, called, critical region, subcritical region, and supercritical region.

10 - All curves have as asymptote, axis OX, and straight line OA.

6.3 - CRITICAL DEPTH. - We have seen that the critical depth is the depth of circulation that makes the content of specific energy, a minimum, for the given discharge. So, that to every discharge corresponds only one critical depth. There, the two alternative levels of energy, are fused into only one level or depth.

In order to find the condition that determines the critical depth, let us derivethe expression of the specific energy in relation to the depth and let us equal such derivation to zero, or let us find the condition for the minimum of specific energy.

In effect:

$$e = d \cos \theta + \alpha \frac{Q^2}{2 g A^2}$$

Deriving (e) in relation to (d), having in mind that A is a function of (d), we will obtain:

$$\frac{d(e)}{d(d)} = \cos \theta - \frac{\alpha Q^2 \; 2 A \; d(A)}{2g \; A^4 \; d(d)} = 0$$

Observing Fig. 6.3, we see that:

$$d(A) = B \, d(d)$$

Substituting this value, in the previous expression, it will give us the condition so that, the discharge Q, passes through the critical depth, or:

$$\cos \theta - \alpha \frac{Q^2 \; B}{g \; A^3} = 0$$

that we can express:

$$\frac{Q^2}{g} = \frac{A_c^3}{B_c} \frac{\cos \theta}{\alpha} \qquad \text{Formula 6.3}$$

The subindex (c), is due to the fact that A_c and B_c, are the values of the area and of the top width when the depth is critical.

We should observe that, A_c as much as B_c, are functions of d_c, so that:

$$Q = f\ (d_c) \qquad \text{or} \qquad d_c = f'\ (Q)$$

If we divide both sides of the formula 6.3, by A_2, we will obtain:

$$\frac{Q^2}{gA_c^2} = \frac{A_c}{B_c} \quad \frac{\cos\ \theta}{\alpha}$$

but:

$$\frac{Q^2}{A_c^2} = V_c \qquad \text{and} \qquad \frac{A_c}{B_c} = d_{mc}$$

therefore:

$$\frac{V_c^2}{g} = d_{mc} \quad \frac{\cos\ \theta}{\alpha}$$

or also:

$$\frac{V_c^2}{g} = \frac{d_{mc}}{2} \quad \frac{\cos\ \theta}{\alpha} \qquad \text{Formula 6.4}$$

The value of the critical velocity will be given by:

$$v_c = \sqrt{g\ d_{mc}}\ \sqrt{\frac{\cos\theta}{\alpha}} \qquad \text{Formula 6.5}$$

The value of the minimum specific energy, will be:

$$e_{min} = d_c \cos\theta + \frac{d_{mc}}{2}\ \frac{\cos\theta}{\alpha} \qquad \text{Formula 6.6}$$

Formula 6.4 can be expressed in the following way:

$$\frac{V_c^2}{g\ d_{mc}} = \frac{\cos\theta}{\alpha}$$

but the first side is the Froude number for the critical conditions or:

$$N_F = \frac{\cos\theta}{\alpha} \qquad \text{Formula 6.7}$$

If in formula 6.2, we clear out Q, we will obtain:

$$Q = \sqrt{\frac{2\ g\ A^2}{\alpha}\ (\ e - d\cos\theta\ )} \qquad \text{Formula 6.8}$$

Equalizing both values, we obtain that:

$$\cos\theta \left(\frac{A}{2B} + d \right) = d \cos\theta + \alpha \frac{Q^2}{2gA^2}$$

$$\cos\theta \frac{A}{2B} = \frac{\alpha Q^2}{2gA^2}$$

Or:

$$\frac{Q^2}{g} = \frac{\cos\theta}{\alpha} \quad \frac{A^3}{B}$$

that isn't anything more than the condition for the discharge Q to pass through the critical depth.

Therefore, the critical depth, can also be defined as the depth, whose discharge is a maximum for a content of constant specific energy.

The experimental studies made in relation to α, indicate that its value is comprehended between 1.03 and 1.36. The value generally is greater in small conduits and lesser in bigger conduits of considerable depth.

When the slope of the conduit is small, the value of angle θ is also small and the cos θ may be considered equal to one. In practice, a slope is considered small if tan θ is equal or less than 1/10 or the 10%. In this case, The graph representation of the previous formula, in a system of axis of rectangular coordinates, that has the discharges as abscisas and the depths as ordinates, has the result of Fig. 6.4.

In effect:

If:

$$d = \frac{e}{\cos \theta}$$, Q is zero and if d = 0, Q, is

also 0. Therefore, between both 0 values of the discharge, should exist a maximum value. The requirement for this, is found deriving the expression 6.8, in relation to (d), and equalizing the derivative to zero. If this is done, this condition is found:

$$2 e \frac{d (A)}{d (d)} - A \cos \theta - 2 d \cos \theta \frac{d (A)}{d (d)} = 0$$

If we remember that d (A)/d (d) = B, top width, the previous expression is reduced to:

$$2 e B = \cos \theta (A + 2 d B)$$

or:

$$e = \cos \theta \left(\frac{A}{2B} + d \right)$$

but the value of the specific energy is given by:

$$e = d \cos \theta + \alpha \frac{Q^2}{2 g A^2}$$

$$\frac{V_c^2}{g\ d_{mc}} = 1 \qquad N_F = 1$$

$$Q = \sqrt{2\ g\ A^2\ (e - d)}$$

6.4.- <u>SUBCRITICAL, CRITICAL AND SUPERCRITICAL FLOW</u>.- The water flows, in free conduits, according to the Froude Number, are classified into:

1. - Subcritical flow
2. - Supercritical flow
3. - Critical flow

In the subcritical flow, the velocity of circulation is inferior to the critical velocity, and the depth of circulation is greater than the critical. The Froude Number is smaller than cos θ/α . So that:

$$V < V_c \qquad d > d_c \qquad N_F < \frac{\cos\ \theta}{\alpha}$$

In the critical flow, the velocity of circulation is equal to the critical and the depth of circulation is equal to the critical depth. The Froude Number is equal to tan Θ is equal to 0.01, Θ = 35", cos Θ = 0.99995, which is practically one.

In practice, in many cases, the values of cos Θ and of α, are usually taken as one. In these cases the previous formulas are reduced to the following:

$$e = d + \frac{V^2}{2\ g}$$

$$e = d + \frac{Q^2}{2\ g\ A^2}$$

$$\frac{Q^2}{g} = \frac{A_c^2}{B_c}$$

$$\frac{V_c^2}{2\ g} = \frac{d_{mc}}{2}$$

$$V_c = \sqrt{g\ d_{mc}}$$

$$e_{min} = \left(d_c + \frac{d_{mc}}{2} \right)$$

circulates with a subcritical depth and at a supercritical velocity. Fig. 6.5 (c).

So, the steady and uniform critical flow has a critical slope.

The steady and uniform subcritical flow has a subcritical slope.

The steady and uniform supercritical flow has a supercritical slope.

6.6. - CRITICAL DEPTH IN CIRCULAR SECTIONS. - Let us suppose that the critical section of a flow is represented in Fig. 6.6. We have seen that the condition to be fulfilled in the critical section is:

$$\frac{Q^2}{g} = \frac{A_c^3}{B_c} \quad \frac{\cos \theta}{\alpha}$$

But in Chapter 2, we have seen that:

$$A = \frac{D^2}{4} \cos^{-1} \left(1 - 2 \frac{d}{D}\right) - \frac{D^2}{2}\left(1 - \frac{2d}{D}\right) \quad \frac{d}{D}\left(1 - \frac{d}{D}\right)$$

$$B = D \sqrt{4 \frac{d}{D}\left(1 - \frac{d}{D}\right)}$$

cos θ/α . So that:

$$V = V_c \qquad d = d_c \qquad N_F = \frac{\cos\theta}{\alpha}$$

In the supercritical flow, the velocity of circulation is superior to the critical velocity, and the depth of ciculation is inferior to the critical depth. The Froude Number is larger than cos θ/α . So that:

$$V > V_c \qquad d > d_c \qquad N_F > \frac{\cos\theta}{\alpha}$$

6.5.- CRITICAL, SUBCRITICAL AND SUPERCRITICAL SLOPE.- The critical slope is that one which should have the bottom of a free conduit, for a discharge that can circulate with a steady uniform flow, at the critical depth. It is designated as s_c , Fig. 6.5 (a).

If the slope of the conduit is inferior to the critical, it is called subcritical slope, and the water circulates with a steady uniform flow, when it circulates with a supercritical depth and at a subcritical velocity. Fig. 6.5 (b).

If the slope of the conduit is superior to the critical, it is said that the slope is supercritical and the water, circulates with a steady uniform flow, when it

Making K_c, equal to the second part of the previous expression, we will have:

$$K_c = \frac{g^{1/2}\left[\frac{1}{4}\cos^{-1}\left(1 - 2\frac{d_c}{D}\right) - \frac{1}{2}\left(1 - \frac{2d_c}{D}\right)\sqrt{\frac{d_c}{D}\left(1 - \frac{d_c}{D}\right)}\right]^{3/2}}{\left[4\frac{d_c}{D}\left(1 - \frac{d_c}{D}\right)\right]^{1/4}}$$

therefore:

$$\frac{Q}{D^{5/2}} = \left[\frac{\cos\theta}{\alpha}\right]^{-1/2} K_c$$

In table 6.1, we give the values of K_c, for different values of d_c/D.

If we want the corresponding discharge at a given critical depth, the value of K_c is calculated, corresponding to d_c/D and then the following formula is applied:

$$Q = \left[\frac{\cos\theta}{\alpha}\right]^{-1/2} K_c D^{5/2}$$

If what we want is the critical depth, we go into the table with the value of K_c, using the formula:

$$K_c = \frac{Q}{D^{5/2}}\left(\frac{\alpha}{\cos\theta}\right)^{1/2}$$

and we look for the corresponding value of d_c/D, from where d_c is cleared out.

If we apply these two formulas that are general, to the critical section and the obtained values are substituted in the expression to be fulfilled in the critical section, we obtain:

$$\frac{Q^2}{g} = \frac{\cos\theta}{\alpha} \frac{\left[\frac{D^2}{4} \cos^{-1}(1-2\frac{d_c}{D}) - \frac{D^2}{2}(1-\frac{2d_c}{D})\sqrt{\frac{d_c}{D}(1-\frac{d_c}{D})}\right]^3}{D\sqrt{4\frac{d_c}{D}(1-\frac{d_c}{D})}}$$

where we can see that the discharge is a function of the critical depth.

The above expression can be written:

$$\frac{Q^2}{g} = \frac{\cos\theta \, D^5 \left[\frac{1}{4}\cos^{-1}(1-2\frac{d_c}{D}) - \frac{1}{2}(1-\frac{2d_c}{D})\sqrt{\frac{d_c}{D}(1-\frac{d_c}{D})}\right]^3}{\alpha\sqrt{4\frac{d_c}{D}(1-\frac{d_c}{D})}}$$

$$Q = \left[\frac{\cos\theta}{\alpha}\right]^{1/2} g^{1/2} D^{5/2} \frac{\left[\frac{1}{4}\cos^{-1}(1-2\frac{d_c}{D}) - \frac{1}{2}(1-\frac{2d_c}{D})\sqrt{\frac{d_c}{D}(1-\frac{d_c}{D})}\right]^{3/2}}{\left[4\frac{d_c}{D}(1-\frac{d_c}{D})\right]^{1/4}}$$

$$\frac{Q}{D^{5/2}} = \left[\frac{\cos\theta}{\alpha}\right]^{1/2} g^{1/2} \frac{\left[\frac{1}{4}\cos^{-1}(1-2\frac{d_c}{D}) - \frac{1}{2}(1-\frac{2d_c}{D})\sqrt{\frac{d_c}{D}(1-\frac{d_c}{D})}\right]^{3/2}}{\left[4\frac{d_c}{D}(1-\frac{d_c}{D})\right]^{1/4}}$$

EQUIVALENT RECTANGULAR SECTION.- Let us suppose that two sections of free conduits, one rectangular and the other circular, are equivalents from the critical point of view, Fig. 6.9. Or, let us suppose that the same discharge circulates on both sections, and let us find the relation between both critical depths.

If we apply the conditional, to obtain the critical depth, to both sections, we obtain:

For the rectangular:

$$\frac{Q^2}{g} = \frac{A_{cr}^3}{B_{cr}} \frac{\cos\theta}{\alpha} \quad o \quad \frac{Q^2}{g} = D^2 d_{cr}^3 \frac{\cos\theta}{\alpha} \quad si\ B = D$$

For the circular:

$$\frac{Q^2}{g} = \frac{A_{cc}^3}{B_{cc}} \frac{\cos\theta}{\alpha}$$

or also:

$$\frac{Q^2}{g} = \frac{\cos\theta}{\alpha} D^5 \frac{\left[\frac{1}{4}\cos^{-1}\left(1-2\frac{d_{cc}}{D}\right) - \frac{1}{2}\left(1-2\frac{d_{cc}}{D}\right)\sqrt{\frac{d_{cc}}{D}\left(1-\frac{d_{cc}}{D}\right)}\right]^3}{\sqrt{4\frac{d_{cc}}{D}\left(1-\frac{d_{cc}}{D}\right)}}$$

If we equal both sides of the final expressions for both sections, we obtain:

$$D^2 d_{cr}^3 = D^5 \frac{\left[\frac{1}{4}\cos^{-1}\left(1-2\frac{d_{cc}}{D}\right) - \frac{1}{2}\left(1-2\frac{d_{cc}}{D}\right)\sqrt{\frac{d_{cc}}{D}\left(1-\frac{d_{cc}}{D}\right)}\right]^3}{\sqrt{4\frac{d_{cc}}{D}\left(1-\frac{d_{cc}}{D}\right)}}$$

$\frac{D_c}{d}$	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0		.0006	.0025	.0055	.0098	.0153	.0220	.0298	.0389	.0491
.1	.0600	.0731	.0866	.1016	.1176	.1347	.1530	.1724	.1928	.2144
.2	.2371	.2609	.2857	.3116	.3386	.3666	.3957	.4259	.4571	.4893
.3	.523	.557	.592	.628	.666	.704	.743	.784	.826	.867
.4	.910	.955	1.000	1.046	1.093	1.141	1.190	1.240	1.291	1.343
.5	1.396	1.449	1.504	1.560	1.616	1.674	1.733	1.793	1.853	1.915
.6	1.977	2.041	2.106	2.172	2.239	2.307	2.376	2.446	2.518	2.591
.7	2.666	2.741	2.819	2.898	2.978	3.061	3.145	3.231	3.320	3.411
.8	3.505	3.603	3.703	3.806	3.914	4.026	4.147	4.272	4.406	4.549
.9	4.70	4.87	5.06	5.27	5.52	5.81	6.19	6.67	7.41	8.83

Table 6.1 - To determine the discharge of a circular conduit partially full, when the circulation is at the critical depth when the discharge is already known.

Another way to obtain the critical depth is by using the graph of Fig. 6.7. There, if we know the value of the abscisa:

$$\frac{Q}{D^{5/2}}\left[\frac{\alpha}{\cos\theta}\right]^{1/2}$$

and the value of the correspondent ordinate, d_c/D, we can obtain the value of d_c.

The critical depth also can be obtained, using the nomogram of Fig. 6.8. This nomogram can be used, also, to calculate the critical depth, for a given discharge that circulates by a rectangular free conduit.

6.7 - CRITICAL DEPTH IN CIRCULAR SECTIONS, BY MEANS OF THE

2. Knowing the value of $2\,d_{cr}/D$ in the graph of Fig. 6.10, we determine the corresponding value of d_{cc}/d_{cr}, which we will designate as E.

3. We determine d_{cc} with the expression:

$$d_{cc} = E\, d_{cr}$$

or also:

$$\left[\frac{d_{cr}}{D}\right]^3 = \frac{\left[\frac{1}{4}\cos^{-1}\left(1-2\frac{d_{cc}}{D}\right) - \frac{1}{2}\left(1-2\frac{d_{cc}}{D}\right)\sqrt{\frac{d_{cc}}{D}\left(1-\frac{d_{cc}}{D}\right)}\right]^3}{\sqrt{4\frac{d_{cc}}{D}\left(1-\frac{d_{cc}}{D}\right)}}$$

what can be written in the following way:

$$128\left(\frac{d_{cr}}{D}\right)^3\left[\frac{d_{cc}}{d_{cr}}\cdot\frac{d_{cr}}{D}\left(1-\frac{d_{cc}}{d_{cr}}\cdot\frac{d_{cr}}{D}\right)\right]^2 =$$

$$\left[\cos^{-1}\left(1-2\frac{d_{cc}}{d_{cr}}\cdot\frac{d_{cr}}{D}\right) - 2\left(1-2\frac{d_{cc}}{d_{cr}}\frac{d_{cr}}{D}\right)\sqrt{\frac{d_{cc}}{d_{cr}}\frac{d_{cr}}{D}\left[1-\frac{d_{cc}}{d_{cr}}\frac{d_{cr}}{D}\right]}\right]^3$$

The previous equation has only two parameters without dimension:

$$\frac{d_{cr}}{D} \qquad \frac{d_{cc}}{d_{cr}}$$

In Fig. 6.10, is represented a graph that establishes the relation between them.

To determine the critical depth of a discharge Q, that circulates through a circular conduit with diameter D, we may procede in the following way:

1. The critical depth for discharge Q, is determined if it would circulate by a rectangular free conduit of width D, and the value of the 2 d_{cr}/D relation is calculated.

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